

Regression with a Misclassified Binary Regressor: Correcting for the Hidden Bias*

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Abstract

We address the estimation of linear regression models with a misclassified binary regressor that is potentially correlated with the other regressors. We show this correlation creates a bias that has been overlooked by existing solutions. This bias arises because the misclassification error is necessarily correlated with the other regressors in the model if the misclassified binary regressor is. It has not shown up in earlier work because it has assumed (explicitly or implicitly) that the misclassified binary variable is orthogonal to other regressors in the model. We show that this ‘hidden’ bias can be substantial and could result in existing estimators taking the wrong sign. We propose two classes of corrections: (i) a bias-adjusted least squares estimator (BALS) that either takes misclassification probabilities as given (e.g. through validation studies) or estimates these probabilities as a first step when a distribution for the true binary regressor is assumed; (ii) parameter bounds that are identified under relatively weak conditions, and do not require any of the above information or assumptions. We prove the consistency and asymptotic normality of the proposed estimators. The finite sample performances of the proposed methods are provided through Monte Carlo simulations, and are compared with existing methods to demonstrate superiority. An empirical application on the effect of food stamp participation on obesity is provided to illustrate the usefulness of the proposed methods in practice.

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Key words: Misclassification, Least Squares, Treatment Effect, Bounds.

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1 Introduction

The use of regression models with one or more binary regressors of interest is common in applied research. A common example is the evaluation of treatment effects. It is well-known that in these models, when the binary regressor is measured with error (i.e. misclassified), the ordinary least squares estimation is inconsistent. However, most of the literature focused on addressing this issue in the simple linear regression setting has had no additional covariates, with the (implicit) assumption that the results would hold in the general case. In the multivariate setting, the issue has been treated almost exclusively with the assumption that the measurement error in the binary regressor is uncorrelated with the other regressors in the model (e.g., [Aigner 1973](#), [Bollinger 1996](#), [Card 1996](#), [Kane, Rouse & Staiger 1999](#), [Black, Berger & Scott 2000](#)). In this paper, we show that in a multivariate regression model this assumption is not innocuous. Any correlation between the misclassified binary regressor and other regressors in the model creates an additional bias that is not corrected by existing estimators. This is because, as we show, any control variable that is correlated with the misclassified binary regressor is necessarily also correlated with the misclassification error and hence with the disturbance term in the operational model, even if this control variable is exogenous in the true (unobserved) model. Since this type of correlation is likely to be common in the data, the associated bias is likely to be frequent in empirical studies. We show that failure to account for this hidden bias can result in severe inconsistencies in parameter estimation including the estimates possibly taking the opposite signs from the true effects.

Many solutions to identify and estimate the parameters of regression models with misclassified binary regressors have been proposed in the literature. A group of papers provide estimators for the model assuming availability of the misclassification probabilities through validation data or other sources (e.g., [Aigner 1973](#), [Freeman 1984](#), [Card 1996](#), [Savoca 2000](#), [Battistin, Nadai & Sianesi 2014](#)). Another group of papers provide solutions based on instrumental variables or repeated measurements (e.g., [Kane et al. 1999](#), [Black et al. 2000](#), [Frazis & Loewenstein 2003](#), [Mahajan 2006](#), [Hu 2008](#), [Hu & Shennach 2008](#), [Battistin et al. 2014](#), [DiTraglia & García-Jimeno 2019](#)). All these studies assume no correlation between the auxiliary regressors (or controls) and the misclassification error in the binary regressor of interest. These methods thus correct for the misclassification bias, but may not correct for the bias stemming from a possible correlation between misclassification error and other regressors in the model (what we term the “hidden bias”). Other works provide parameter bounds when additional information about misclassification probabilities or extraneous information are not available (e.g., [Klepper 1988](#), [Bollinger 1996](#), [Kreider & Pepper 2007](#), [Kreider 2010](#), [Kreider, Pepper, Gunder-](#)

sen & Jolliffe 2012, van Hasselt & Bollinger 2012, Bollinger & van Hasselt 2017). More recently, Nguimkeu, Denteh & Tchernis (2019) analyzed the bias in both ordinary least squares (OLS) and instrumental variable (IV) estimators when the measurement error is endogenous and depends on one or more covariates. They proposed a two-step consistent estimator; however, their method only corrects for the biases when the misclassification error is unidirectional.

In this paper, we propose two classes of corrections. The first builds from Aigner (1973)'s method and proposes a bias-adjusted least squares estimator (BALS) that takes misclassification probabilities as given. It expresses the correlation between misclassification error and control variables as a function of quantities that can be obtained through sample statistics, and uses it to correct for the bias in the least squares estimator. The modified least squares estimator (MLS) developed by Aigner (1973) is a special case of our estimator when both the misclassified binary regressor and the associated measurement error are uncorrelated with other regressors in the model. This type of estimator is often motivated by the presence of misclassification probabilities that can generally be obtained through validation data (e.g., Aigner 1973, Freeman 1984, Card 1996, Savoca 2000, Battistin et al. 2014, Courtemanche, Denteh & Tchernis 2019). When these probabilities are not directly available, we exploit the maximum likelihood estimator of Hausman, Abrevaya & Scott-Morton (1998) to derive the misclassification probabilities, and then use them in our BALS formula. The BALS estimates of the parameters of the outcome model with this modification are also consistent if the distribution for the true binary regressor is correctly assumed. The second approach that we propose extends the procedure in van Hasselt & Bollinger (2012) to multivariate models and provides bounds for the model parameters that are identified under relatively weak conditions, without assuming knowledge of the misclassification probabilities or of the distribution of the true binary regressor. We prove the \sqrt{n} -consistency and asymptotic normality of the proposed estimators. Monte Carlo simulations results are provided to show the finite sample performance of our proposed methods under various misclassification rates in the binary regressor and to demonstrate their superiority over existing methods. The proposed procedures are applied to real data to estimate the effect of food stamp participation on body mass index.

Our contribution is threefold. First, we believe the proposed BALS estimator is the first to provide consistent point estimates of a multivariate linear regression model with a mismeasured binary regressor that is correlated with other regressors, when misclassification rates are available. Second, when these rates are not available and the distribution of the true binary regressor is correctly assumed, our two-step estimator is a useful alternative to existing ones such as Brachet (2008)

and [Almada, McCarthy & Tchernis \(2016\)](#). Third, we provide new parametric bounds that are tighter than the [Bollinger \(1996\)](#)'s bounds and can be used when there is scant or no knowledge of the misclassification rates or their distribution. The paper proceeds as follows. In Section 2, we reanalyze the properties of the OLS and MLS estimators in a multivariate linear regression model with a misclassified binary regressor, paying a particular attention to the potential correlation between the misclassification error and the control variables. In Section 3, we develop a bias-adjusted least squares (BALS) estimator, its two-step version, and the parameter bounds. We also establish their large sample properties, including their consistency and asymptotic normality. Section 4 provides Monte Carlo simulations. Section 5 illustrates the proposed methods in an empirical example. Section 6 summarizes our findings. Mathematical proofs are in the appendix.

2 Framework

We consider a multiple linear regression model, for observation i in a random sample of size n , given by

$$Y_i = c + \beta D_i^* + X_i' \gamma + \varepsilon_i, \quad \mathbb{E}[\varepsilon_i | X_i, D_i^*] = 0, \quad (1)$$

where Y_i is the scalar outcome variable, X_i is a $k \times 1$ vector of correctly measured regressors, $D_i^* \in \{0, 1\}$ is a binary regressor (or dummy variable) with $\Pr[D_i^* = 1] = P^*$, and $P^* \in (0, 1)$. The error term ε_i is assumed mean zero and uncorrelated with X_i and D_i^* . The purpose is to estimate the model parameters c , β and γ .

In the treatment effect literature, D_i^* is the treatment status or program participation status, X_i is the vector of control variables, and β captures the treatment effect, which is of great interest for program evaluation (for a review, see, e.g., [Abadie & Cattaneo 2018](#)). The econometrician does not observe the true status D_i^* but a potentially misclassified binary surrogate D_i . Specifically, instead of observing D_i^* , we observe D_i such that

$$D_i = D_i^* + U_i \quad (2)$$

where U_i denotes a measurement error, independent of ε_i , with $U_i \in \{-1, 0, 1\}$, and $\Pr[D_i = 1] = P$, with $P \in (0, 1)$. Because D_i^* is binary, it is necessarily correlated (negatively) with the measurement error, U_i . Hence, the measurement error in the observed binary variable D_i is nonclassical. Since the true participation status, D_i^* , is unobserved and only the surrogate D_i is observed, the equivalent model with reported participation status estimated by the researcher, usually referred to as the *operational model*, is given by

$$Y_i = c + \beta D_i + X_i' \gamma + (\varepsilon_i - \beta U_i). \quad (3)$$

The nonclassical nature of the measurement error in D_i makes estimation of the model parameters difficult.

Regression models with a misclassified binary regressor such as (1)-(3) have been studied in many papers. The first was [Aigner \(1973\)](#) who showed not only that the ordinary least squares (OLS) estimator for the coefficient, β , of the misclassified binary regressor is underestimated, but also that the OLS estimators of the other coefficients, γ , of the correctly measured independent variables are inconsistent as well. [Aigner \(1973\)](#) proposed a modified least squares (MLS) estimator to consistently estimate the parameters of this model when the misclassification probabilities are known. [Savoca \(2000\)](#) extends [Aigner \(1973\)](#)'s analysis to the case when several binary regressors are misclassified. [Black et al. \(2000\)](#) and [Kane et al. \(1999\)](#) show that when repeated misclassified measurements of the binary regressor are available, one can obtain consistent estimates of the model using the generalized method of moments (GMM).

However, the corrections provided by [Aigner \(1973\)](#)'s MLS and other existing methods are satisfactory only to the extent that the correctly measured covariates X_i are asymptotically uncorrelated with the true binary regressor D_i^* . In fact, given the exogeneity assumption in the true model, the bias in the coefficients of the correctly measured regressors has been viewed merely as a contamination from the bias in the estimated coefficient of the misclassified variable D_i . Therefore, the bias correction strategy in [Aigner \(1973\)](#) and related papers relied exclusively on accounting for the correlation between the mismeasured variable D_i and the error term $(\varepsilon_i - \beta U_i)$ in the operational model. However, as we show below, if a misclassified binary regressor D_i^* in the linear regression model is correlated with the correctly measured regressors X_i , then these regressors must also be correlated with the measurement error U_i , and hence the operational model error $(\varepsilon_i - \beta U_i)$, even if these variables were initially exogeneous. A more general solution should not focus only on solving the ‘‘endogeneity’’ of D_i in the operational model (3); it must also consider the ‘‘endogeneity’’ of X_i in this model. Otherwise, only part of the bias in the coefficients of the misclassified binary regressor as well as in the correctly measured regressors would be eliminated, except under limited conditions. This missing link has not been noticed before and as a consequence the findings of some of the previous works on this issue suffer from a ‘‘hidden bias’’. We analyze the effect of this hidden bias on OLS and MLS estimators.

Examples of linear regression models where the binary regressor D_i^* and the

correctly measured regressors X_i are correlated are commonly encountered in empirical studies. In the food stamp and obesity model of [Almada et al. \(2016\)](#), D_i^* is the true food stamp participation status, D_i is the self-reported food stamp participation, and X_i is a vector of covariates including age, household size, number of children, mother's education, employment status, marital status, gender and race. These covariates are correlated with true participation status as later found by [Courtemanche et al. \(2019\)](#) using administrative records as true measures of food stamp participation. In the model of technology adoption and agricultural productivity estimated by [Wossen, Abdoulaye, Alene, Nguimkeu, Feleke, Rabbi, Haile & Manyong \(2019\)](#), D_i^* is true adoption status of improved cassava varieties (obtained through DNA-fingerprinting technology) versus landraces, and D_i is the self-reported adoption status by responding farmers. They found that D_i^* is significantly correlated with components of X_i , which included regressors such as age, education, mobile phone ownership, membership in a cassava growers association, and whether or not a friend or neighbor is a true adopter. Another popular example is in the return to education model such as [Kane & Rouse \(1995\)](#). Here, the true D_i^* is a binary indicator of obtaining a college degree, while D_i is the self-reported indicator of college graduation. The vector of correctly measured covariates consists of the respondent's age, gender, race, and family income, all of which [Black, Sanders & Taylor \(2003\)](#) found to be significantly correlated with measurement of higher education in the census data. These few examples are far from being exhaustive, but they illustrate that in many empirical situations the misclassified binary regressor of interest is likely to be correlated with other regressors of the model, and further justifies the relevance of the issues addressed in this paper for the empiricist.

We first discuss the estimation of the slope parameters, (β, γ) , of the model before looking at the estimation of the intercept parameter, c , which is rather easy to derive once the former are obtained.

Assumption 1. $\text{Var}(X_i)$ exists and is nonsingular.

This assumption is standard and rules out perfect multicollinearity among the columns of X_i and the extreme cases where some of the components of X_i may be following an ill-behave distribution (e.g. distributions with no finite moments). Under Assumption 1, the probability limit of the ordinary least squares (OLS) estimator of β and γ from the operational model (3) are given by

$$\text{plim} \begin{bmatrix} \hat{\beta}_{OLS} - \beta \\ \hat{\gamma}_{OLS} - \gamma \end{bmatrix} = -\beta \begin{bmatrix} \text{Var}(D_i) & \text{Cov}(D_i, X_i) \\ \text{Cov}(X_i, D_i) & \text{Var}(X_i) \end{bmatrix}^{-1} \begin{bmatrix} \text{Cov}(D_i, U_i) \\ \text{Cov}(X_i, U_i) \end{bmatrix} \quad (4)$$

This result describes the nature of the bias in the OLS estimators when misclassification is present. In particular, the OLS bias has two components: the first

component is driven by $\text{Cov}(D_i, U_i)$ which arises directly from the presence of misclassification errors, i.e., $\text{Cov}(D_i, U_i) \neq 0$. The second component is a little more subtle and depends on whether $\text{Cov}(X_i, U_i)$ is zero or not. As explained earlier, the existing literature largely assumed that $\text{Cov}(X_i, U_i)$ is zero, leaving only the first component of the bias as the subject of interest. However, as we show below, in a multiple linear regression model where $\text{Cov}(D_i^*, X_i) \neq 0$, we cannot overlook the second component of the bias because $\text{Cov}(X_i, U_i)$ is a direct function of $\text{Cov}(X_i, D_i^*)$. To further understand the structure of these relationships, we impose the following restriction, which is common in many related papers.

We define the probabilities of false positives and false negatives by

$$\Pr[D_i = 1|X_i, D_i^* = 0] = \alpha_0 \quad \text{and} \quad \Pr[D_i = 0|X_i, D_i^* = 1] = \alpha_1 \quad (5)$$

These probabilities are assumed conditionally constant, implying that the misclassification probabilities are uncorrelated with X and ε , conditionally on the true response. This assumption implies that the measurement error may vary with the covariates, but only through the true response, and has been made in several papers, (e.g., [Aigner 1973](#), [Bollinger 1996](#), [Bollinger & David 1997](#), [Kreider & Pepper 2007](#), [Bollinger & van Hasselt 2017](#), [van Hasselt & Bollinger 2012](#), [Black et al. 2000](#), [Kane et al. 1999](#), [Hausman et al. 1998](#)).¹ [Meyer & Mittag \(2017\)](#) refer to this condition as the misclassification probabilities being *conditionally random*. We maintain this assumption partly because modifying it would require to modify the standard results with which the proposed estimators are compared, but also because assuming that misclassification probabilities vary directly with the regressors brings other important complications that are beyond the scope of this paper.² We also make the following assumption which is standard in the literature and is often referred to as the *monotonicity assumption*.

Assumption 2. $\alpha_0 + \alpha_1 < 1$

This condition is equivalent to $\text{Cov}(D_i, D_i^*) > 0$, and ensures that in spite of the presence of misclassification error in the responses, the reported status D_i is still an informative proxy for true status D_i^* . If this is not the case, then the measurement error is so severe that $1 - D_i$ would be a better measure of D_i^* than is D_i . We have the following result which is key to our subsequent discussions.

Lemma 1. *Let U_i be the misclassification error associated with D_i^* in the reporting equation given by (2) and let X_i be any vector of variables. Then*

$$\text{Cov}(X_i, U_i) = -(\alpha_0 + \alpha_1) \text{Cov}(X_i, D_i^*). \quad (6)$$

¹A simple example could be the case where the reported treatment is defined by $D_i = D_i^* + U_i$, with $U_i = a_0 - (a_0 + a_1)D_i^*$, where $a_j = \mathbf{1}[u_i < \alpha_j]$, $j \in \{0, 1\}$, and $u \sim \mathcal{U}(0, 1)$.

²This includes, for instance, dimensionality issues in identifying the misclassification probabilities at the individual level.

This result establishes the relationship between $\text{Cov}(X_i, D_i^*)$ and $\text{Cov}(X_i, U_i)$ and shows that the second quantity is always nonzero if the first quantity is. We note that when there is no misclassification error (i.e., $\alpha_0 = \alpha_1 = 0$) we have $\text{Cov}(D_i^*, U_i) = 0$ so that the OLS estimator is unbiased and consistent.

Corollary 1. *Consider the linear regression model (1) subject to nonclassical measurement error defined by Equation (2). If $\text{Cov}(X_i, D_i^*) \neq 0$, then $\text{Cov}(X_i, \varepsilon_i - \beta U_i) \neq 0, \forall X_i$.*

This result says that if the true binary regressor is correlated with the correctly measured control variables in a multiple linear regression model, then both the reported binary regressor and the correctly measured controls are endogenous in the operational model given by Equation (3), even if they are exogenous in the true (unobserved) model given by Equation (1). This result follows directly from Lemma 1. Since this result holds in general, including for arbitrary variables that are not necessarily part of the model, it also provides an important insight for why linear instrumental variable estimation methods can not correct for the bias generated by misclassification. Any instrument variable Z that is correlated with the misclassified regressor D_i^* (i.e. Z is relevant) would, by virtue of Corollary 1, be also systematically correlated with the disturbance term in the operational model (i.e. Z cannot be exogenous), and hence would be an invalid instrument for the IV method.

3 Correcting the Bias of the OLS Estimator

We propose three types of estimators that correct for the OLS bias. The first assumes knowledge of misclassification probabilities from outside information, the second estimates these probabilities given the sample observations and a distribution for the true binary regressor, and the third estimates the model parameter bounds without such knowledge or assumptions.

3.1 The Bias-Adjusted Least Squares Estimator

The results given by Equations (4) suggest that with consistent estimators for $\text{Cov}(D_i, U_i)$ and $\text{Cov}(X_i, U_i)$ we can readily correct for the OLS biases by inverting the terms on the right hand side of these equations around β and γ . The next lemma, which follows from Lemma 1 above and Equation (3c) of Aigner (1973), shows that these two components, indeed, can be expressed in terms of observables under the assumptions stated above.

Lemma 2. Under Assumptions 1 and 2,

$$\text{Cov}(D_i, U_i) = \zeta \text{Var}(D_i) \quad \text{and} \quad \text{Cov}(X_i, U_i) = -\theta \text{Cov}(X_i, D_i) \quad (7)$$

$$\text{where } \theta = \frac{\alpha_0 + \alpha_1}{1 - \alpha_0 - \alpha_1} \quad \text{and} \quad \zeta = 1 - \frac{(P - \alpha_0)(1 - \alpha_1 - P)}{(1 - \alpha_0 - \alpha_1)(1 - P)P}$$

This result shows that the two components $\text{Cov}(X_i, U_i)$ and $\text{Cov}(D_i, U_i)$ driving the OLS asymptotic bias given by Equations (4) can be estimated using sample statistics if misclassification probabilities α_0 and α_1 are known. The remaining components of this asymptotic bias can as usual be estimated using sample statistics. We summarize that in the following theorem.

Theorem 1. Under Assumptions 1-2, the asymptotic biases of the OLS estimators of β and γ are given by

$$\text{plim} \begin{bmatrix} \hat{\beta}_{OLS} - \beta \\ \hat{\gamma}_{OLS} - \gamma \end{bmatrix} = \beta \begin{bmatrix} \text{Var}(D_i) & \text{Cov}(D_i, X_i) \\ \text{Cov}(X_i, D_i) & \text{Var}(X_i) \end{bmatrix}^{-1} \begin{bmatrix} -\zeta \text{Var}(D_i) \\ \theta \text{Cov}(X_i, D_i) \end{bmatrix} \quad (8)$$

where ζ and θ dependent on α_0, α_1 as given in Lemma 2.

Equations 8 summarizes the structure of the biases in the OLS estimation of Model (3) under the above assumptions. When $\text{Cov}(D_i, X_i) = 0$ or if there are no auxiliary covariates in the model, then $\hat{\gamma}_{OLS}$ is consistent for γ , but $\text{plim } \hat{\beta}_{OLS} = \beta \left(1 - \frac{\text{Cov}(D_i, U_i)}{\text{Var}(D_i)}\right) = \beta(1 - \zeta)$ as obtained by Aigner (1973). Given knowledge of ζ , Aigner (1973) then proposed a modified least squares estimator (MLS) to correct for the OLS bias which consists in dividing the OLS estimator by the proportionate bias. If, however, $\text{Cov}(D_i, X_i) \neq 0$ as would be the case if $\text{Cov}(D_i^*, X_i) \neq 0$, then both the OLS and the MLS estimators of β and γ would be inconsistent due to the terms that involve θ in the right hand side of Equations 8. These terms generate a bias that has been largely overlooked in the literature, and we therefore refer to it as the *hidden bias*. Interestingly, this component does not induce only a downward (or attenuation bias) for the OLS, but could lead to a sign-reversal bias for both OLS and MLS. This is particularly obvious for the estimation of γ . To see this, suppose that β is positive, and notice that we can write

$$\text{plim } \hat{\gamma}_{OLS} = \gamma + \beta (\theta + \zeta) \Delta^{-1} \text{Cov}(X_i, D_i) \quad (9)$$

$$\text{and} \quad \text{plim } \hat{\gamma}_{MLS} = \gamma + \beta \zeta \Delta^{-1} \text{Cov}(X_i, D_i)$$

with $\Delta = \text{Var}(X_i) - \text{Cov}(X_i, D_i)\text{Var}(D_i)^{-1}\text{Cov}(D_i, X_i)$. Then for sufficiently high levels of misreporting and positive correlation between X_i and D_i^* , $\text{plim } \hat{\gamma}_{OLS}$ could be positive, even if γ is negative. As an example, in the return to college model,

if college graduation is strongly correlated with Female and misreporting is high, then the OLS and MLS estimates could wrongly conclude that women earn more than men on average (a contradiction to the well-known wage-gap in labor economics). Likewise, in the technology adoption and farm productivity model, if the true adoption status is strongly correlated with plot size and misreporting of adoption status is high, then OLS and MLS estimates could wrongly conclude that marginal increases in plot size increase productivity (a contradiction to the well-known farm size - productivity inverse relationship in agricultural economics).

The above results can now be used to propose consistent estimators of the model. Denote by S_{VW} the sample covariance between V and W , that is, $S_{VW} = \frac{1}{n} \sum_{i=1}^n (V - \bar{V})(W - \bar{W})'$, where \bar{V} and \bar{W} are the sample means of V and W , respectively. We have the following result.

Theorem 2 (Bias Adjusted Least Squares). *Let Assumptions 1-2 hold. For given misclassification probabilities α_0 and α_1 , define the bias-adjusted least squares estimators by*

$$\begin{bmatrix} \hat{\beta} \\ \hat{\gamma} \end{bmatrix} = \begin{bmatrix} (1 - \zeta)S_{DD} & S_{DX} \\ (1 + \theta)S_{XD} & S_{XX} \end{bmatrix}^{-1} \begin{bmatrix} S_{YD} \\ S_{YX} \end{bmatrix} \quad (10)$$

where ζ and θ depend on α_0, α_1 as given in Lemma 2. Then:

- (i) These estimators are consistent, i.e. $\hat{\beta} \xrightarrow{p} \beta$ and $\hat{\gamma} \xrightarrow{p} \gamma$.
- (ii) If, in addition, $\mathbb{E}[\varepsilon_i^2 | X_i, D_i^*] = \sigma^2$, then these estimators are asymptotically normal, i.e. $\sqrt{n} \begin{bmatrix} \hat{\beta} - \beta \\ \hat{\gamma} - \gamma \end{bmatrix} \xrightarrow{d} \mathcal{N}(\mathbf{0}, \Omega^{-1} \Xi \Omega^{-1})$, where the matrices Ω , and Ξ are given by Equations (20) and (23) in the appendix.

We term this estimator the bias-adjusted least squares (BALS) estimator. We think of it as an adjustment of the ordinary least squares estimator because it can be written as a linear combination of the OLS estimator as follows

$$\begin{bmatrix} \hat{\beta} \\ \hat{\gamma} \end{bmatrix} = \begin{bmatrix} \hat{\beta}_{OLS} \\ \hat{\gamma}_{OLS} \end{bmatrix} + \hat{\beta}_{OLS} \begin{bmatrix} (1 - \zeta)S_{DD} & S_{DX} \\ (1 + \theta)S_{XD} & S_{XX} \end{bmatrix}^{-1} \begin{bmatrix} \zeta S_{DD} \\ -\theta S_{XD} \end{bmatrix}. \quad (11)$$

On the other hand, the modified least squares estimators can also be written in terms of the OLS as

$$\begin{bmatrix} \hat{\beta}_{MLS} \\ \hat{\gamma}_{MLS} \end{bmatrix} = \begin{bmatrix} \hat{\beta}_{OLS} \\ \hat{\gamma}_{OLS} \end{bmatrix} + \hat{\beta}_{OLS} \begin{bmatrix} (1 - \zeta)S_{DD} & S_{DX} \\ S_{XD} & S_{XX} \end{bmatrix}^{-1} \begin{bmatrix} \zeta S_{DD} \\ 0 \end{bmatrix}. \quad (12)$$

Put in these forms, it is easy to notice that when there is no measurement error, that is, if $\alpha_0 = \alpha_1 = 0$, then we have $\theta = 0, \zeta = 0$, so that BALS, OLS and MLS

are equal, i.e., $\hat{\beta} = \hat{\beta}_{MLS} = \hat{\beta}_{OLS}$ and $\hat{\gamma} = \hat{\gamma}_{MLS} = \hat{\gamma}_{OLS}$ and are all consistent. When there is misclassification error and X and D^* are uncorrelated, the proposed estimator, BALS, and Aigner (1973)'s MLS estimator are both consistent and asymptotically equivalent, but OLS is inconsistent. Finally, when there is misclassification error and X and D^* are correlated, only the proposed estimator, BALS, is consistent while OLS and MLS are inconsistent. Once β and γ have been consistently estimated, a consistent estimator for the intercept, c , can be obtained as follows.³

Corollary 2. *Under the conditions of Theorem 1, define the bias-adjusted least squares estimator of the intercept, c , by*

$$\hat{c} = \bar{Y} - \hat{\beta} \frac{\bar{D} - \alpha_0}{1 - \alpha_0 - \alpha_1} - \bar{X}' \hat{\gamma}. \quad (13)$$

Then \hat{c} is consistent and asymptotically normal, i.e. $\hat{c} \xrightarrow{p} c$ and $\sqrt{n}(\hat{c} - c) \xrightarrow{d} \mathcal{N}(0, \sigma_c^2)$, where the asymptotic variance, σ_c^2 , is given by Equation (28) in the appendix.

Note that the homoscedasticity condition assumed in part (ii) of Theorem 2 is not needed to establish the asymptotic normality of the proposed estimators. These estimators would still be asymptotically normal even if the variance of the error terms in the true model has an unknown form. It is only needed to derive asymptotic variance for these estimators that are practical and usable. These asymptotic variances (as given in the appendix) can be consistently estimated by simply plugging in sample information and can be used to perform standard inference on the regression coefficients. If one assumes heteroskedasticity of a given form, then one should modify these expressions to account for the desired error covariance structure. Heteroskedasticity of an unknown form would however require a more sophisticated device, such as a nonparametric conditional variance estimation (see, e.g., Ruppert, Wand, Holst & Hossjer 1997, Fan & Yao 1998)

3.2 The Case of Unknown Misclassification Rates

A limitation of the bias-adjusted estimator developed above is the required knowledge of misclassification probabilities α_0 and α_1 . This limitation is shared with all methods that require upfront knowledge of these probabilities (e.g., Aigner 1973, Freeman 1984, Card 1996, Savoca 2000, Battistin et al. 2014) A way around it is to use a two-step process, first estimating the requisite probabilities and then

³Alternatively, the BALS estimator can be estimated including the intercept term directly as $\begin{bmatrix} (1 - \alpha_1)\mathbf{i}'\mathbf{D} & D'\mathbf{X} \\ (1 + \theta)\mathbf{X}'D & \mathbf{X}'\mathbf{X} \end{bmatrix}^{-1} \begin{bmatrix} D'\mathbf{X} \\ \mathbf{X}'Y \end{bmatrix}$, where $\mathbf{D} = D - \alpha_0\mathbf{i}$, $\mathbf{X} = [\mathbf{i} \ X]$ and \mathbf{i} is an n -vector of 1's. We treat the intercept separately throughout for easy comparison with previous results.

using them in the BALS formula. With a distribution (or functional form) for the conditional probability of the true binary regressor, i.e., $\Pr[D^* = 1|X]$, one can estimate the misclassification probabilities α_0 and α_1 by maximum likelihood or nonlinear least squares as proposed by Hausman et al. (1998). Those estimates can then be used in Equation (10) to obtain feasible versions of the BALS that are also consistent under correct specification of the binary regressor's distribution.

The starting point of the first step in this case is a parametric model for the true binary variable D_i^* . It assumes that D_i^* follows the model

$$\Pr[D_i^* = 1|X_i] = F(X_i'\pi), \quad (14)$$

where $F(\cdot)$ is a known, strictly increasing, cumulative distribution function (CDF) and π is a vector of parameters. Since $\Pr[D_i = 1|X_i] = \alpha_0 + (1 - \alpha_0 - \alpha_1) \Pr[D_i^*|X_i]$, the model for the reported binary regressor is then given by

$$\Pr[D_i = 1|X_i] = \alpha_0 + (1 - \alpha_0 - \alpha_1)F(X_i'\pi)$$

The fact the binary regression model and the outcome equation have the same set of regressors X_i is not especially problematic because of the nonlinearity and the fact that any factor driving D_i^* that we think belongs to the outcome equation should be included in X_i as well. However, the specification of the binary choice probability may account for the possibility of i.i.d. unobserved heterogeneity that are excluded from the outcome equation. The maximum likelihood (ML) estimators of α_0 , α_1 and π , denoted $\hat{\alpha}_0$, $\hat{\alpha}_1$, $\hat{\pi}$ can be obtained by maximizing the likelihood function

$$\begin{aligned} \mathcal{L}(\alpha_0, \alpha_1, \pi) = \frac{1}{n} \sum_i^n \{ & D_i \ln(\alpha_0 + (1 - \alpha_0 - \alpha_1)F(X_i'\pi)) \\ & + (1 - D_i) \ln(1 - \alpha_0 - (1 - \alpha_0 - \alpha_1)F(X_i'\pi)) \} \end{aligned} \quad (15)$$

with respect to $(\alpha_0, \alpha_1, \pi)$. Hausman et al. (1998) showed that under Assumptions 1- 2 and correct specification of $F(\cdot)$, these estimators are consistent.

Theorem 3 (Bias Adjusted Least Squares: Two-Step). *Let the conditions of Theorem 1 hold, and assume the function $F(\cdot)$ is correctly specified and strictly increasing (i.e. everywhere positive density). Denote by $\tilde{\beta}$ and $\tilde{\gamma}$ the estimators derived from Equations (10) where α_0 and α_1 are replaced by their estimates $\hat{\alpha}_0$ and $\hat{\alpha}_1$ from the above step. Then*

- (i) *These estimators are consistent, i.e. $\tilde{\beta} \xrightarrow{p} \beta$ and $\tilde{\gamma} \xrightarrow{p} \gamma$.*
- (ii) *If, in addition, $\mathbb{E}[\varepsilon_i^2|X_i, D_i^*] = \sigma^2$, then these estimators are asymptotically normal, i.e. $\sqrt{n} \begin{bmatrix} \tilde{\beta} - \beta \\ \tilde{\gamma} - \gamma \end{bmatrix} \xrightarrow{d} \mathcal{N}(\mathbf{0}, \Omega^{-1}\Xi\Omega^{-1})$, where the asymptotic variance components Ω and Ξ are the same as in Theorem 1.*

The two-step estimation of c follows as in Corollary 1, by using the slopes estimates $(\tilde{\beta}, \tilde{\gamma})$ and the estimated misclassification probabilities $\hat{\alpha}_0$ and $\hat{\alpha}_1$ obtained from the above steps in Equation (13). The resulting estimator, denoted \tilde{c} is consistent and asymptotically normal, with the same asymptotic variance as \hat{c} .

As stated above, the consistency of the misclassification probabilities depends on correct specification of the first-stage equation. If there is doubt about the functional form chosen to estimate the probability $\Pr[D_i^* = 1|X_i]$, semiparametric procedures which do not require distributional assumptions can be used to estimate these probabilities. Some of these methods include the semiparametric ML of Hausman et al. (1998) based on the maximum rank correlation estimator of Han (1987), or the maximum score estimator of Manski (1985). The main drawback of nonparametric or semiparametric methods is that their local nature yields estimators of misclassification rates that are slower than \sqrt{n} - rate of convergence, potentially yielding inflated variance or inconsistency for estimators of the parameters in the outcome model. Nonetheless, semiparametric approaches can at least serve as a specification check for the first step estimation.

It is important to note that another two-step estimation procedure for the model in this framework was advocated by Brachet (2008) while estimating the relationship between error-ridden maternal smoking status and birthweight. The first step consisted of using the procedure in Hausman et al. (1998) to estimate the true probability of smoking (rather than the misclassification probabilities as we do here), which is then used in the outcome equation in lieu of the true binary regressor to estimate the model. Almada et al. (2016) used a similar approach in a panel data setting to estimate the effect of error-ridden food stamp participation on obesity. In both cases, however, the consistency and asymptotic normality of their estimators were not formally established.

3.3 Bounding the Parameters of the Model

The above discussions assumed knowledge of either the misclassification probabilities or the distribution of the true binary regressor. A useful alternative to these approaches is to bound parameter estimates (e.g. Klepper 1988, Bollinger 1996). For the remainder of the paper, we assume, without any loss of generality, that $\beta \geq 0$, or equivalently, that $\text{Cov}(D^*, Y|X) \geq 0$. As a starting point, it is useful to notice that from our derivations above, we can write:

$$\beta = bB(\alpha_0, \alpha_1)$$

where, for the ease of notation, all estimators are understood in terms of their probability limit, and we denote $b = \text{plim } \widehat{\beta}_{OLS}$, and

$$B(\alpha_0, \alpha_1) = \frac{(1 - \alpha_0 - \alpha_1)P(1 - P)(1 - R_{DX}^2)}{(P - \alpha_0)(1 - \alpha_1 - P) - R_{DX}^2P(1 - P)},$$

where $R_{DX}^2 = \frac{\text{Cov}(D, X)\text{Var}(X)^{-1}\text{Cov}(X, D)}{\text{Var}(D)}$ is the R-squared of the linear regression of D_i on X_i . Since $B(\alpha_0, \alpha_1)$ is increasing in both α_0 and α_1 , the upper bound (respectively, the lower bound) for β is attained when α_0 and α_1 take their maximum values (respectively, minimum values), given Assumption 2. The lower bound is obtained when the misclassification probabilities are zero, that is, $B(0, 0) = 1$. This implies that the parameter β can be bound as follows

$$b \leq \beta \leq bB(\alpha_0^u, \alpha_1^u) \quad (16)$$

where α_0^u and α_1^u are the upper bounds of the misclassification probabilities, which may sometimes be obtained from outside sources even when the exact misclassification rates are unavailable or unclear. Then, using Equation (9), the components of the parameter vector γ can be bounded between γ_{OLS} and $\gamma_{OLS} - b(\zeta^u + \theta^u)B(\alpha_0^u, \alpha_1^u)\Delta^{-1}\text{Cov}(X_i, D_i)$, where ζ^u and θ^u depend on α_0^u, α_1^u by the formula given in Lemma 2, and the lower bound and upper bound of each component of γ is determined by the sign of each corresponding component in the vector $\Delta^{-1}\text{Cov}(X_i, D_i)$. The bounds of c can then similarly be deduced from the bounds of β and γ , using Equation (13).

However, if the upper bounds of the misclassification probabilities are not available, the procedure that we develop below can be used to gain insight about the desired estimates. In what follows, we propose a method to bound the parameters β , γ and c in the model defined by Equation (1), given the data available. The bounding method we propose is a generalization of [van Hasselt & Bollinger \(2012\)](#) to a multiple linear regression setting where the binary regressor is possibly correlated with control variables. By projecting all the variables of the model, including the misclassification error, to the column space of the control variables, we can “partial-out” the effect of these regressors on the binary regressor and the misclassification error. The residual model is then a simple linear regression model with a structure similar to the one discussed in [van Hasselt & Bollinger \(2012\)](#). Hence, the bounds derived by these authors in the simple linear regression model with a single binary regressor can be modified and applied to the residual model to bound β , and then γ and c .

For the ease of the exposition, we denote for any variable V_i , its residual $\tilde{V}_i = V_i - \mathbb{L}[V_i|X_i]$, where $\mathbb{L}[V_i|X_i]$ is the linear projection of V_i on X_i . Then, taking the difference between the variables and their linear projections on X_i , we can derive a residual model for Equation (1) by:

$$\tilde{Y}_i = \beta \tilde{D}_i^* + \varepsilon_i, \quad \mathbb{E}[\varepsilon_i | \tilde{D}_i^*] = 0 \quad (17)$$

We also use the following notations, where, again, all the stated estimators are in terms of their probability limits. Note that b is the slope of the OLS estimator of the regression of \tilde{Y}_i on \tilde{D}_i , i.e., we can write $b = \sigma_{\tilde{D}\tilde{Y}} / \sigma_{\tilde{D}}^2$, where, $\sigma_{\tilde{D}\tilde{Y}} = \text{Cov}(\tilde{D}, \tilde{Y})$ and $\sigma_{\tilde{D}}^2 = \text{Var}(\tilde{D})$; let $\psi = [\psi_1, \dots, \psi_k]'$ and ψ_0 denote, respectively, the slope and intercept of the regression of Y_i on X_i i.e., $\psi = \text{Var}(X_i)^{-1} \text{Cov}(X_i, Y_i)$ and $\psi_0 = E[Y_i] - E[X_i]' \psi$; let $\lambda = [\lambda_1, \dots, \lambda_k]'$, and λ_0 denote, respectively, the slope and the intercept of the linear regression of D_i on X_i , i.e. $\lambda = \text{Var}(X_i)^{-1} \text{Cov}(X_i, D_i)$, and $\lambda_0 = E[D_i] - E[X_i]' \lambda$; and let $\sigma_j^2 = \text{Var}[Y_i | X_i, D_i = j]$ denote the conditional variance of Y_i given X and $D_i = j$, for $j = 0, 1$. We have the following results.

Theorem 4. *Let Assumptions 1-2 hold. Assume, without loss of generality, that $\beta > 0$, and define the parameters κ_0 , κ_1 and κ by*

$$\kappa_0 = (1 - P) \frac{\sigma_0^2}{\sigma_{\tilde{D}\tilde{Y}}}, \quad \kappa_1 = P \frac{\sigma_1^2}{\sigma_{\tilde{D}\tilde{Y}}}, \quad \text{and} \quad \kappa = \kappa_0 \mathbf{1}[P > 1/2] + \kappa_1 \mathbf{1}[P \leq 1/2].$$

Then:

(i) *The parameter β is bounded as follows.*

$$b \leq \beta \leq \begin{cases} b + \kappa_0 [P + (1 - P)R_{DX}^2] & \text{for } P > 1/2 \\ b + \kappa_1 [(1 - P) + PR_{DX}^2] & \text{for } P \leq 1/2 \end{cases}$$

(ii) *The components of γ are bounded by the terms $\psi - b\lambda$ and $\psi - (b + \kappa)\lambda$, where lower bound and upper bound of each of the components of γ are determined by the sign of each corresponding component in λ .*

(iii) *The intercept, c , is bounded between $\min\{\psi_0 - b\lambda_0, \psi_0 - (b + \kappa)\lambda_0\}$ and $\max\{\psi_0 - b\lambda_0, \psi_0 - (b + \kappa_1)\lambda_0 + \kappa_1 P(1 - R_{DX}^2)\}$.*

(iv) *The misclassification probabilities are bounded as follows:*

$$0 \leq \alpha_0 \leq \frac{P^2 \sigma_1^2 (1 - R_{DX}^2)}{b \sigma_{\tilde{D}\tilde{Y}} + P \sigma_1^2}; \quad 0 \leq \alpha_1 \leq \frac{(1 - P)^2 \sigma_0^2 (1 - R_{DX}^2)}{b \sigma_{\tilde{D}\tilde{Y}} + (1 - P) \sigma_0^2}.$$

These bounds are sharper than the bounds derived in [Bollinger \(1996\)](#). They can be easily estimated consistently by taking their sample counterparts, and their asymptotic variances can be computed using the usual delta method so that standard inference on these bounds can be performed.

4 Monte Carlo Simulations

We assess the finite sample performance of our estimators through Monte Carlo simulations, comparing the proposed bias adjusted estimators with the OLS and the MLS estimators, and the proposed parameter bounds with the Bollinger bounds. Our goal is to consistently estimate the parameters (c, β, γ') of the model given by Equation (1), assuming that true binary response D_i^* is unobserved, but a misclassified surrogate D_i and the control variables X_i are observed.

4.1 Simulation setup

The data generating process is simulated as follows. The exogenous covariates $X_i = (X_{1i}, X_{2i})$ are generated by $X_{1i} = z_{1i}^2$ and $X_{2i} = z_{2i}$, where $\begin{pmatrix} z_{1i} \\ z_{2i} \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 & 0.3 \\ 0.3 & 1 \end{pmatrix}\right)$. Denote $\pi \in \{-0.8; 1.2\}$, and define the true binary regressor, D_i^* , by

$$D_i^* = \mathbf{1}[\pi_0 + \pi_1 X_{1i} + \pi_2 X_{2i} - u_i \geq 0], \quad u_i \sim \mathcal{N}(0, 1),$$

where $\pi_0 = -\text{sign}(\pi)$ (i.e. 1 when $\pi = -0.8$, and -1 when $\pi = 1.2$), $\pi_1 = \pi$, and $\pi_2 = 0.9\pi$. We consider alternative values of the parameter π to examine the performance of the estimators under varying degrees of the correlation between the true binary regressor D_i^* and the covariates $X_i = (X_{1i}, X_{2i})$, which translates to a correlation between misclassification error and covariates as implied by Lemma 1. The outcome equation Y_i is given by

$$Y_i = c + \beta D_i^* + \gamma_1 X_{1i} + \gamma_2 X_{2i} + \epsilon_i \quad \text{where } \epsilon_i \sim \mathcal{N}(0, 2),$$

and $c = 1$; $\beta = 4$; $\gamma_1 = -0.3$; $\gamma_2 = 0.2$. are the true population regression parameters we seek to estimate.

The econometrician does not have the above model at hand, but only an operational model defined by

$$Y_i = c + \beta D_i + \gamma_1 X_{1i} + \gamma_2 X_{2i} + \varepsilon_i,$$

where the observed binary regressor D_i is an error-ridden one defined by

$$D_i = D_i^* \mathbf{1}(v_i > \alpha_1) + (1 - D_i^*) \mathbf{1}(v_i < \alpha_0),$$

with the disturbance $v_i \sim \mathcal{U}(0, 1)$ drawn from a uniform distribution, and the misclassification probabilities given by $\alpha_0, \alpha_1 \in [0, 1)$ such that $\alpha_0 + \alpha_1 < 1$. We

consider the following set of values for the misclassification probabilities $(\alpha_0, \alpha_1) \in \{(0, 0.15); (0, 0.30); (0.15, 0.15); (0.15, 0.30); (0.30, 0.15)\}$. This allows us to assess the performance of the proposed estimators for increasing rates of misclassification, under all three possible cases of misclassification errors: one-sided misclassification $(\alpha_0, \alpha_1) \in \{(0, 0.15); (0, 0.30)\}$, symmetric misclassification $(\alpha_0, \alpha_1) \in \{(0.15, 0.15)\}$, and asymmetric misclassification $(\alpha_0, \alpha_1) \in \{(0.15, 0.30); (0.3, 0.15)\}$.

The point-estimates of the model parameters $c, \beta, \gamma_1, \gamma_2$ are obtained using our proposed bias adjusted least square estimator (BALS) which we compare with the OLS estimator and the modified least squares estimator (MLS). For the OLS we report both the estimates based on the true (unobserved) binary regressor and the estimates using the misclassified (observed) binary regressor. The reported MLS estimates only use known misclassification rates, and the proposed BALS estimates are reported for both the known misclassification rates (Known α 's) and the estimated misclassification rates using the [Hausman et al. \(1998\)](#) approach assuming the true rates are unavailable (Unknown α 's). The lower bounds and the upper bounds of the model parameters are also reported for $c, \beta, \gamma_1, \gamma_2$, as well as for the misclassification rates α_0 and α_1 , using both the [Bollinger \(1996\)](#) approach and our proposed approach described in the previous section.

4.2 Simulation Results for Point Estimates

For each of the parameter cases, we executed 1000 replications, each using a sample size of $n = 5000$ observations. Table 1 reports Monte Carlo simulation results for OLS, MLS and the proposed BALS estimators. The OLS estimates based on the true binary regressor is given in the column denoted 'True' as a benchmark, whereas the column denoted 'Observed' gives the naive OLS estimates based on the observed data. The results show that not only the OLS estimators of $\beta, \gamma_1, \gamma_2$ and c are inconsistent as shown in Theorem 1, but the inconsistency of the estimates of γ_1 and/or γ_2 can sometimes produce wrong signs. Sign-switching in these estimates especially occurred when false negatives and/or false positives rates are high (e.g. $\alpha_1 \geq 30\%$ or $\alpha_0 \geq 15\%$) and the direction of the correlation between the covariate and the true binary regressor, π , is opposite from the true covariate effect, $\gamma_j, j = 1, 2$. The modified least squares estimates are reported in the column named 'MLS'. The results show that the MLS estimates are also inconsistent but usually perform much better than the OLS. As one would expect, the largest inconsistencies in the MLS estimates come from the covariates coefficients γ_1, γ_2 , while the inconsistency in β are somewhat moderate.

As discussed in the theory, sign-switching sometimes also occur in the MLS estimates of γ_1 and γ_2 , especially when, as with the OLS cases, misclassification

Table 1: Simulations Results for Point Estimates

α_0	α_1	π	Para.	True Values	OLS		MLS	BALS	
					True	Observed		Known	Unknown
0%	15%	-0.8	β	4.0	4.001	2.863	3.731	4.003	4.020
			γ_1	-0.3	-0.299	-0.539	-0.426	-0.299	-0.296
			γ_2	0.2	0.200	-0.082	0.051	0.200	0.204
			c	1.0	0.999	2.169	-	0.998	0.980
		1.2	β	4.0	4.002	3.021	3.653	4.007	4.000
			γ_1	-0.3	-0.300	-0.079	-0.162	-0.302	-0.300
	30%	-0.8	β	4.0	4.004	2.226	3.495	4.007	4.032
			γ_1	-0.3	-0.299	-0.674	-0.538	-0.298	-0.293
			γ_2	0.2	0.201	-0.238	-0.079	0.201	0.206
			c	1.0	0.996	2.822	-	0.993	0.966
		1.2	β	4.0	4.001	2.424	3.359	4.012	4.013
			γ_1	-0.3	-0.300	0.058	-0.044	-0.301	-0.301
15%	15%	-0.8	β	4.0	4.000	2.198	3.497	4.010	4.045
			γ_1	-0.3	-0.301	-0.678	-0.539	-0.299	-0.292
			γ_2	0.2	0.199	-0.242	-0.079	0.202	0.210
			c	1.0	0.999	2.507	-	0.992	0.955
		1.2	β	4.0	3.999	2.105	3.352	4.001	4.013
			γ_1	-0.3	-0.300	0.091	-0.044	-0.300	-0.302
	30%	-0.8	β	4.0	4.001	1.587	3.295	4.027	4.039
			γ_1	-0.3	-0.300	-0.779	-0.635	-0.295	-0.293
			γ_2	0.2	0.200	-0.360	-0.191	0.209	0.211
			c	1.0	0.998	3.095	-	0.978	0.964
		1.2	β	4.0	4.002	1.567	3.103	4.009	4.006
			γ_1	-0.3	-0.301	0.187	0.056	-0.301	-0.300
30%	-0.8	β	4.0	4.004	1.708	3.304	4.042	4.024	
		γ_1	-0.3	-0.299	-0.767	-0.632	-0.29	-0.295	
		γ_2	0.2	0.201	-0.348	-0.190	0.211	0.206	
		c	1.0	0.996	2.770	-	0.963	0.976	
	1.2	β	4.0	4.003	1.548	3.108	4.016	4.033	
		γ_1	-0.3	-0.301	0.189	0.056	-0.303	-0.306	
30%	-0.8	β	4.0	4.004	1.708	3.304	4.042	4.024	
		γ_1	-0.3	-0.299	-0.767	-0.632	-0.29	-0.295	
		γ_2	0.2	0.201	-0.348	-0.190	0.211	0.206	
		c	1.0	0.996	2.770	-	0.963	0.976	
	1.2	β	4.0	4.003	1.548	3.108	4.016	4.033	
		γ_1	-0.3	-0.301	0.189	0.056	-0.303	-0.306	
30%	-0.8	β	4.0	4.004	1.708	3.304	4.042	4.024	
		γ_1	-0.3	-0.299	-0.767	-0.632	-0.29	-0.295	
		γ_2	0.2	0.201	-0.348	-0.190	0.211	0.206	
		c	1.0	0.996	2.770	-	0.963	0.976	
	1.2	β	4.0	4.003	1.548	3.108	4.016	4.033	
		γ_1	-0.3	-0.301	0.189	0.056	-0.303	-0.306	

These are simulations results with 1000 replications and 5000 observations, where $(\alpha_0, \alpha_1) \in \{(0, .15); (0, .3); (.15, .15); (.15, .30); (.3, .15)\}$ and dependence between D^* and X is captured by $\pi \in \{-0.8; 1.2\}$.

rates are high and the direction of the correlation between the true binary regressor and the covariates is opposite to true covariate effect. This can be seen, for instance, for the results of γ_1 estimation when $\alpha_0 = 15\%$, $\alpha_1 = 30\%$ and $\pi = 1.2$, or the results of γ_2 estimation when $\pi = -0.8$ within the same scenario. When these inconsistencies, including possible sign-switching, occur in empirical studies with real data, they could have dramatic consequences for evidence-based policy. In contrast, the proposed estimator (BALS), reported in the last two columns of Table 1, gives consistent estimates of the true model parameters and are superior to both the naive OLS and MLS estimators. The BALS estimates using both known misclassification probabilities (see panel ‘Known’ of BALS) or using estimated misclassification probabilities (see panel ‘Unknown’ of BALS) give results that are quantitatively similar to those that would obtain from the OLS using the true (unobserved) binary regressor (see panel ‘True’ of OLS). Importantly, the BALS estimates appear to be insensitive to the degree of misclassification in the data.

4.3 Simulation Results for Parameter Bounds

This set of Monte Carlo simulations estimates parameter bounds assuming we know neither the misclassification rates nor the distribution of the true binary regressor. We run 1000 replications each using a sample size of $n = 5000$. Simulation results are presented in Table 2. Our benchmark are the bounds from Bollinger (1996) who was the first to propose a method to estimate parametric bounds for the multiple linear regression framework discussed in this paper, under weak assumptions about the misclassification rates. The lower and upper bounds for the model parameters $\beta, \gamma_1, \gamma_2, c$ and the misclassification probabilities α_0, α_1 using Bollinger (1996)’s approach are presented in the first panel of Table 2. Our proposed lower and upper bounds are given in the last two panels of the table. Our bounds are tighter than the Bollinger bounds for all the parameters considered. In particular, for each of the slope coefficients $\beta, \gamma_1, \gamma_2$, one of our bounds is always identical to Bollinger whereas the other is always tighter than the corresponding Bollinger bound. This is also true for the misclassification probabilities where only the upper bounds are estimated. For the intercept, both our lower and upper bounds are tighter than the Bollinger bounds. In addition, our proposed bounds also preserve the sign of the true coefficient more often than the Bollinger bounds. Overall, the parameter ranges from our bounds are between 45% and 57% of the ranges obtained using Bollinger’s approach.

We now look at how these methods perform in a real data application.

Table 2: Simulations Results for Parameter Bounds

π	Parameters	True Value	Bollinger		Proposed	
			Lower Bound	Upper Bound	Lower Bound	Upper Bound
-0.8	β	4.0	2.863	8.483	2.863	6.063
	γ_1	-0.3	-0.539	0.641	-0.539	0.133
	γ_2	0.2	-0.082	1.302	-0.082	0.706
	c	1.0	-3.587	2.169	-1.111	2.169
	α_0	0.00	0	0.296	0	0.229
	α_1	0.15	0	0.291	0	0.254
1.2	β	4.0	3.021	9.718	3.021	6.054
	γ_1	-0.3	-1.322	-0.079	-0.642	-0.079
	γ_2	0.2	-1.269	0.520	-0.290	0.520
	c	1.0	-1.152	1.583	0.281	1.520
	α_0	0.00	0	0.220	0	0.170
	α_1	0.15	0	0.317	0	0.279
-0.8	β	4.0	2.198	10.011	2.198	6.074
	γ_1	-0.3	-0.677	0.643	-0.677	-0.022
	γ_2	0.2	-0.241	1.307	-0.241	0.527
	c	1.0	-5.780	2.508	-1.604	2.508
	α_0	0.15	0	0.404	0	0.357
	α_1	0.15	0	0.310	0	0.269
1.2	β	4.0	2.104	10.07	2.104	6.295
	γ_1	-0.3	-1.321	0.092	-0.657	0.092
	γ_2	0.2	-1.273	0.763	-0.317	0.763
	c	1.0	-3.441	1.701	-1.117	1.611
	α_0	0.15	0	0.339	0	0.302
	α_1	0.15	0	0.356	0	0.313
-0.8	β	4.0	1.587	11.196	1.587	5.685
	γ_1	-0.3	-0.780	0.642	-0.780	-0.173
	γ_2	0.2	-0.361	1.304	-0.361	0.349
	c	1.0	-6.363	3.096	-0.939	3.096
	α_0	0.15	0	0.393	0	0.353
	α_1	0.30	0	0.433	0	0.407
1.2	β	4.0	1.568	12.815	1.568	6.411
	γ_1	-0.3	-1.326	0.185	-0.466	0.185
	γ_2	0.2	-1.266	0.899	-0.033	0.899
	c	1.0	-4.043	2.406	-0.721	2.057
	α_0	0.15	0	0.335	0	0.302
	α_1	0.30	0	0.471	0	0.445

These are simulations results with 1000 replications and 5000 observations, where $(\alpha_0, \alpha_1) \in \{(0, 0.15); (0.15, 15); (0.15, 0.30)\}$ and dependence between D^* and X is captured by $\pi \in \{-0.8; 1.2\}$.

5 The Effect of SNAP on BMI

This section illustrates the applicability of the methods developed in the previous sections to estimate the effect of participation in the Supplemental Nutrition Assistance Program (SNAP) on Body Mass Index (BMI), where the participation indicator may be misclassified. The data are from the public-use version of the FoodAPS survey and are well described in [Courtemanche et al. \(2019\)](#). We assume that apart from SNAP participation the other variables are correctly measured and we ignore the potential endogeneity of SNAP participation in order to focus our attention on the methods derived in the preceding sections. Our objective is to compute the proposed BALS estimates and the proposed parameter bounds, as well as the OLS, MLS and Bollinger bounds for comparison.

The Supplemental Nutrition Assistance Program (SNAP) is a federal nutrition program which provides nutrition benefits worth more than 60 billion US dollars a year to supplement the food budget of over 35 million individuals so that they can purchase healthy food and move towards self-sufficiency. Misclassification errors in SNAP participation status recorded in national surveys is well documented and the false negative rate estimated in some studies are as high as almost 50% (see, e.g., [Meyer, Mittag & Goerge 2018](#)). FoodAPS is a nationally representative survey of US households that measures household food purchases as well as health and nutrition outcomes with a survey and linked administrative data on SNAP. Using these data, [Courtemanche et al. \(2019\)](#) conducted a validation study and constructed 12 different “gold standards” and associated misclassification rates in SNAP participation by combining the information available in two linked administrative data sources (see [Courtemanche et al. 2019, Table 4](#)). To illustrate our methods and better capture how severe the issues discussed in this paper could be in practice, we pick the gold standard with the highest misclassification probabilities in the list provided by [Courtemanche et al. \(2019\)](#). Specifically, their “ADMIN alternate 1” gold standard that we are using gives a false negatives rate of 32.31% and a false positives rate of 4.78%. We start by taking these misclassification rates as given, and we use them in our proposed method to estimate the effect of SNAP on BMI using the self-reported survey data while controlling for other factors. The summary statistics of the variables use are presented in [Table 3](#) and gives the mean, the standard deviation as well as the pairwise correlation of each of the variables with the SNAP dummy, the error-ridden binary regressor of interest. In our estimations, we use the full set of control variables given in [Table 3](#), but we only report a selected set whose correlation with the SNAP indicator is particularly high or particularly low, in order to focus our attention on the issues

Table 3: Summary Statistics

Variable	Mean	Std. Dev.	Correlation with SNAP
BMI	27.920	6.870	0.099
SNAP	0.353	0.478	1.000
Age (years)	42.705	17.891	-0.125
White	0.683	0.465	-0.083
Black	0.147	0.354	0.139
Female	0.538	0.499	0.053
Married	0.425	0.494	-0.200
Previously married	0.213	0.409	0.109
GED and above	0.313	0.464	0.081
Some college	0.280	0.449	-0.058
Bachelor's degree or higher	0.180	0.384	-0.226
Employed	0.502	0.500	-0.171
Family income (monthly, 1000\$)	3.890	3.824	-0.326
Household size	3.282	1.919	0.163
Rural	0.266	0.442	-0.009
Distance to primary food store (miles)	3.263	4.918	-0.015
Authorized primary food store	0.976	0.154	0.007

relevant to our earlier theoretical discussion.⁴ In particular, as can be seen in the last column of Table 3, variables such as Female, Rural and Distance to primary food store have relatively low pairwise correlation with SNAP (with correlation coefficients 0.053, -0.009, and -0.015 respectively), whereas variables such as Bachelor's degree or higher, Family income, and Household size have relatively higher pairwise correlation with SNAP (with correlation coefficients -0.226, -0.326, and 0.163 respectively). The other regressors (unreported in our estimation results) have correlations with either similar magnitudes or somewhere in between. Since the severity of the hidden bias highlighted in this paper is associated with the extent of the correlation between the misclassified binary regressor and the controls variables, it seems useful to particularly look at these extreme correlation cases.

Table 4 presents the model parameter estimates with OLS, MLS and BALS for the given misclassification probabilities of $(\alpha_0, \alpha_1) = (0.0478, 0.3231)$ taken from the validation study of Courtemanche et al. (2019). There is a clear discrep-

⁴The full set of controls used in our estimations include Age, White, Black, Female, Married, Previously married, GED and above, Some college, Bachelor's degree or higher, Employed, Family income, Household size, Rural, Distance to primary food store, Authorized primary food store, and the full set of estimations are available from the authors.

Table 4: Results using Misclassification Rates from Validation Data

Variable	Dependent variable: BMI		
	OLS	MLS	BALS
SNAP	1.155 *** (0.164)	1.924 *** (0.350)	2.651*** (0.279)
Female	0.097 (0.142)	0.082 (0.142)	0.083 (0.147)
Bachelor’s degree or above	-1.185 *** (0.241)	-1.062*** (0.239)	-0.693 *** (0.251)
Family income (monthly, 1000\$)	-0.137*** (0.021)	-0.110*** (0.020)	-0.032 (0.023)
Household size	0.064 (0.042)	0.024 (0.041)	-0.094** (0.045)
Rural	-0.199 (0.180)	-0.201 (0.181)	-0.207 (0.186)
Distance to primary food store (miles)	0.050 *** (0.016)	0.050 *** (0.016)	0.050*** (0.016)
Constant	24.155 *** (0.572)	- -	22.933*** (6.532)

Notes: Standard errors are in parentheses. Significance codes: ‘*’ $p < 0.1$, ‘**’ $p < 0.05$, ‘***’ $p < 0.01$. The misclassification probabilities from validation data are $(\alpha_0, \alpha_1) = (0.0478, 0.3231)$ and are taken from Table 4 ADMIN Alternate 1 in Courtemanche et al. (2019). Regressors not reported include Age, White, Black, Married, Previously married, GED and above, Some college, Employed, Authorized primary food store.

ancy between all three methods presented, as they show completely different and sometimes opposite results. The effect of SNAP on BMI estimated with BALS is 2.651, that is about 40% higher than the estimate obtained with MLS, and more than twice the estimate obtained with OLS. The estimated coefficients on the selected set of control variables display similar differences across the three different methods. While these discrepancies exist regardless of the degree of correlation between the controls and the misclassified binary regressor (implied by the mere presence of measurement errors as already known in the literature), they tend to be exacerbated with higher levels of correlation (as we emphasize in this paper). For example, while the estimated coefficients for the relatively lowly correlated controls (Female, Rural, and Distance to primary food store) are not very different across methods or are insignificant otherwise, the differences in the estimated coefficients of the relatively highly correlated controls (Bachelor’s degree or higher, Family income, and Household size) are more substantial across various methods. Interestingly, there is a sign-switch in the effect of Household size. While the effect

of the latter is positive in the OLS and MLS estimates, it is negative and significant in the BALS estimates. As explained in the theory, this may arise because in this application the given misclassification probabilities are high and the correlation between Household size and SNAP participation is also relatively high and positive (as shown in Table 3). As a result, the bias component is high and positive, leaving the OLS and MLS estimates of the effect of Household size on BMI positive as obtained here, even if the true effect could be negative as suggested by our BALS estimate. As for the variables Bachelor's or higher and Family income, although they are both also relatively highly correlated with SNAP in the data, their estimated effects from OLS and MLS likely would not switch sign since these correlations are negative as are their expected effects on BMI.

Table 5: Results using Estimated Misclassification Rates

Variable	OLS	MLS	BALS
SNAP	1.155*** (0.164)	1.557 (1.086)	1.756*** (0.222)
Female	0.097 (0.142)	0.090 (0.142)	0.076 (0.143)
Bachelor's degree or higher	-1.185*** (0.241)	-1.121*** (0.239)	-1.004*** (0.244)
Family income (monthly, 1000\$)	-0.137*** (0.021)	-0.123*** (0.020)	-0.098*** (0.022)
Household size	0.064 (0.042)	0.043 (0.041)	0.006 (0.043)
Rural	-0.199 (0.180)	-0.200 (0.180)	-0.202 (0.182)
Distance to primary food store miles)	0.050*** (0.016)	0.050*** (0.016)	0.050*** (0.016)
Constant	24.155*** (0.573)	–	23.756*** (6.686)

Notes: Standard errors are in parentheses. Significance codes: '*' $p < 0.1$, '**' $p < 0.05$, '***' $p < 0.01$. The misclassification probabilities are $(\alpha_0, \alpha_1) = (0.0553, 0.1761)$ and are estimated using the parametric procedure of Hausman et al. (1998). Regressors not reported include Age, White, Black, Married, Previously married, GED and above, Some college, Employed, Authorized primary food store.

Given that the proposed BALS estimator uses misclassification probabilities as inputs and that these probabilities are not always available, it is useful to explore alternate approaches for obtaining them. This motivates the use of a method such as the Hausman et al. (1998) procedure for obtaining these misclassification prob-

abilities as a possible first step in applying the proposed BALS. Table 5 reports the model parameter estimates with OLS, MLS and BALS based on the estimated misclassification probabilities obtained from the Hausman et al. (1998) procedure using the full set of regressors and assuming normality.⁵ The false positives and false negatives rates derived from this procedure are 5.53% and 17.61%, respectively. While the estimated false negative rate is lower than the one used in Table 4, its value still lies within the range of gold standards provided in Courtemanche et al. (2019). Likewise, the estimated false positives rate obtained here is not too different from the one used in Table 4 and lies within the range of gold standards in Courtemanche et al. (2019). The results in Table 5 also shows discrepancies between the three methods and confirm that biases could be substantial. The level of discrepancies is however lower than in Table 4 given that the rate of false negatives utilized here is much lower.

Table 6: Estimates of Parameter Bounds

Variable	Proposed (BALS)		Proposed (Bounds)		Bollinger Bounds	
	Known	Unknown	Lower	Upper	Lower	Upper
SNAP	2.651	1.756	1.155	66.392	1.155	158.241
Female	0.083	0.076	-1.645	0.097	-4.098	0.097
Bachelor or above	-0.693	-1.004	-1.185	13.318	-1.185	33.738
Family income	-0.032	-0.098	-0.137	2.947	-0.137	7.288
Household size	-0.094	0.006	-4.590	0.064	-11.142	0.064
Rural	-0.207	-0.202	-0.437	-0.199	-0.772	-0.199
Distance to food store	0.050	0.050	0.042	0.050	0.029	0.050
Constant	22.933	23.756	-17.806	24.155	-76.886	24.155
False positives rate α_0	0.048	0.055	0.000	0.273	0.000	0.275
False negatives rate α_1	0.323	0.176	0.000	0.501	0.000	0.503

Notes: The left panel of the column Proposed (BALS) denoted ‘Known’ reports the results of BALS using the misclassification rates from validation study as given in Table 4. The right panel denoted ‘Unknown’ reports the results of BALS using the estimated misclassification rates as given in Table 5.

Finally, Table 6 present our parameters bounds when the misclassification probabilities are not given, and the distribution of the true binary regressor is not assumed. The BALS estimates in Tables 4 and 5 are added for comparison; they are given in panel ‘Known’ (for the case where α_0 and α_1 are known from validation studies) and ‘Unknown’ (for the case where α_0 and α_1 are unknown but estimated

⁵We do not present the full results of the first-step estimation from Hausman et al. (1998) procedure since the only output needed there are the estimated misclassification rates; but they are available from the authors for the interested reader.

using [Hausman et al. 1998](#)), respectively, under the column ‘Proposed (BALS)’ in [Table 6](#). In addition, we also report parameter bounds based on [Bollinger \(1996\)](#) which confirm that our bounds are tighter. While the estimates from the proposed bounds are tighter than the Bollinger bounds, they both cover both the BALS estimates based on validation misclassification rates and the BALS estimates based on estimated misclassification rates. This is also true for the bounds obtained for the misclassification probabilities. The proposed bounds are tighter than the Bollinger bounds and both cover the misclassification rates obtained from the validation study of [Courtemanche et al. \(2019\)](#) as well as the estimates obtained from the [Hausman et al. \(1998\)](#) procedure.

6 Concluding Remarks

The existing literature on the least-squares estimation of the multivariate linear regression model with misclassified binary regressors has ignored an important component of the underlying bias which potentially affects parameter estimates of all regressors. This bias is driven by the correlation that may exist between the misclassification error in the binary regressor and other model regressors in empirical studies. As a result, estimation procedures such as the modified least squares (MLS) estimator proposed by [Aigner \(1973\)](#) are asymptotically biased unless the misclassified binary regressor is orthogonal to all other regressors. This bias has been carried over subsequent related work and methods on this issue. We propose two classes of corrections. The first is a bias adjusted least squares (BALS) estimator that can directly use misclassification probabilities when they are known or available (e.g. through validation studies). This estimator corrects for the OLS bias and we show that it is \sqrt{n} -consistent and asymptotically normal. When misclassification probabilities are unknown or unavailable, we show that the proposed BALS estimator performs equally well when these probabilities are estimated using the [Hausman et al. \(1998\)](#) procedure, provided a correct distribution for the true binary regressor is assumed. The second is a bounding method that extends the [van Hasselt & Bollinger \(2012\)](#) approach to the multiple linear regression setting and provides bounds that are tighter than the [Bollinger \(1996\)](#) bounds. This approach provides a useful insight for partially identifying the parameters of a multivariate linear regression model with misclassification errors, especially when the misclassification probabilities are unknown and a distribution for the true binary regressor is not assumed. Monte Carlo simulations clearly demonstrate the perverse effect of the hidden bias and show how both the proposed BALS estimator and the proposed parameter bounds perform well and are superior to existing methods such as OLS, [Aigner \(1973\)](#)’s MLS and [Bollinger \(1996\)](#)’s bounds. As illustrated from an empirical example estimating the effect of

SNAP on BMI while controlling for other relevant variables that are potentially correlated with SNAP, the differences between the proposed methods and existing methods can be substantial. The proposed estimators are easily implementable and can be used in many other applied settings.

The methods proposed in this paper do not, however, deal with the issue of endogeneity of the misclassified binary regressor, which is an important one in practice especially for studies involving non-experimental data. Our main focus is on correcting for the hidden bias that exist and could be severe even in the case where the true binary regressor is exogenous. Another limitation is the assumption of constant misclassification rates that we maintain throughout our analysis. This limitation may be substantial in applications where the measurement error is endogenous and vary with covariates beyond the true binary regressor (e.g., [Kreider et al. 2012](#), [Nguimkeu et al. 2019](#)). These considerations are left for future research.

7 Appendix: Mathematical Proofs

7.1 Proof of Lemma 1

Proof. We can write:

$$\text{Cov}(X_i, U_i) = \mathbb{E}[X_i U_i] - \mathbb{E}[X_i] \mathbb{E}[U_i] = \mathbb{E}[X_i \mathbb{E}[U_i | X_i, D_i^*]] - \mathbb{E}[X_i] \mathbb{E}[\mathbb{E}[U_i | X_i, D_i^*]],$$

where the last display follows from the Law of Iterated Expectations (L.I.E). Now,

$$\mathbb{E}[U_i | X_i, D_i^* = 0] = \Pr[U_i = 1 | X_i, D_i^* = 0] = \Pr[D_i = 1 | X_i, D_i^* = 0] = \alpha_0$$

$$\mathbb{E}[U_i | X_i, D_i^* = 1] = -\Pr[U_i = -1 | X_i, D_i^* = 1] = -\Pr[D_i = 0 | X_i, D_i^* = 1] = -\alpha_1.$$

Hence, $\mathbb{E}[U_i | X_i, D_i^*] = \alpha_0(1 - D_i^*) - \alpha_1 D_i^* = \alpha_0 - (\alpha_0 + \alpha_1) D_i^*$. It follows that $\mathbb{E}[X_i U_i] = \alpha_0 \mathbb{E}[X_i] - (\alpha_0 + \alpha_1) \mathbb{E}[X_i D_i^*]$ and $\mathbb{E}[U_i] = \alpha_0 - (\alpha_0 + \alpha_1) \mathbb{E}[D_i^*]$, so that $\text{Cov}(X_i, U_i) = -(\alpha_0 + \alpha_1) (\mathbb{E}[X_i D_i^*] - \mathbb{E}[X_i] \mathbb{E}[D_i^*]) = -(\alpha_0 + \alpha_1) \text{Cov}(X_i, D_i^*)$. \square

7.2 Proof of Corollary 1

Proof. Since $\text{Cov}(X_i, \varepsilon_i) = 0$ by assumption, then

$$\text{Cov}(X_i, \varepsilon_i - \beta U_i) = -\beta \text{Cov}(X_i, U_i) = \beta(\alpha_0 + \alpha_1) \text{Cov}(X_i, D_i^*)$$

where the last equality follows from Lemma 1. If $\text{Cov}(X_i, D_i^*) \neq 0$, then we must have $\text{Cov}(X_i, \varepsilon_i - \beta U_i) \neq 0$, provided we are not in a trivial situation where β is zero or the misclassification probabilities are both zero. \square

7.3 Proof of Lemma 2

Proof. For the first equality, we already know from Lemma 1 that $\text{Cov}(X_i, U_i) = -(\alpha_0 + \alpha_1)\text{Cov}(X_i, D_i^*) = -(\alpha_0 + \alpha_1)\text{Cov}(X_i, \mathbb{E}[D_i^*|X_i])$, by the L.I.E. Now, since

$$\begin{aligned}\mathbb{E}[D_i|X_i] &= \Pr[D_i = 1|D_i^* = 0] \Pr[D_i^* = 0|X_i] + \Pr[D_i = 1|D_i^* = 1] \Pr[D_i^* = 1|X_i] \\ &= \Pr[D_i = 1|D_i^* = 0](1 - \Pr[D_i^* = 1|X_i]) + (1 - \Pr[D_i = 0|D_i^* = 1]) \Pr[D_i^* = 1|X_i] \\ &= \alpha_0 + (1 - \alpha_0 - \alpha_1)\mathbb{E}[D_i^*|X_i]\end{aligned}$$

This means

$$\mathbb{E}[D_i^*|X_i] = \frac{\mathbb{E}[D_i|X_i] - \alpha_0}{1 - \alpha_0 - \alpha_1} \quad (18)$$

It follows that

$$\text{Cov}(X_i, U_i) = -(\alpha_0 + \alpha_1)\text{Cov}\left(X_i, \frac{\mathbb{E}[D_i|X_i] - \alpha_0}{1 - \alpha_0 - \alpha_1}\right) = -\left(\frac{\alpha_0 + \alpha_1}{1 - \alpha_0 - \alpha_1}\right)\text{Cov}(X_i, D_i)$$

The second equality follows from Equation (3c) of [Aigner \(1973\)](#), which states that $\text{Cov}(D_i, U_i) = (\eta + \nu)\text{Var}(D_i)$, where $\eta = \Pr[D_i^* = 0|D_i = 1] = \alpha_0 \frac{\Pr[D_i^* = 0]}{\Pr[D_i = 1]} = \alpha_0 \frac{1 - P^*}{P}$ and $\nu = \Pr[D_i^* = 1|D_i = 0] = \alpha_1 \frac{\Pr[D_i^* = 1]}{\Pr[D_i = 0]} = \alpha_1 \frac{P^*}{1 - P}$. Notice that Equation (18) above implies $P^* = \frac{P - \alpha_0}{1 - \alpha_0 - \alpha_1}$. It follows that

$$\eta = \alpha_0 \frac{1 - P^*}{P} = \alpha_0 \frac{1 - \alpha_1 - P}{(1 - \alpha_0 - \alpha_1)P}, \quad \nu = \alpha_1 \frac{P - \alpha_0}{(1 - \alpha_0 - \alpha_1)(1 - P)}.$$

If we denote $\zeta = \eta + \nu = 1 - \frac{(P - \alpha_0)(1 - \alpha_1 - P)}{(1 - \alpha_0 - \alpha_1)P(1 - P)}$, we then have $\text{Cov}(D_i, U_i) = \zeta\text{Var}(D_i)$, the desired result. \square

7.4 Proof of Theorem 1

Proof. The proof is straightforward. It follows from taking Equation (4) and plugging in the values of $\text{Cov}(D_i, U_i)$ and $\text{Cov}(X_i, U_i)$ given by the identities established in Lemma 2. \square

7.5 Proof of Theorem 2

Proof. (i) Consistency. The equation set which determines $(\hat{\beta}, \hat{\gamma})$ is given by

$$\begin{bmatrix} S_{DY} \\ S_{XY} \end{bmatrix} = \begin{bmatrix} (1 - \zeta)S_{DD} & S_{DX} \\ (1 + \theta)S_{XD} & S_{XX} \end{bmatrix} \begin{bmatrix} \hat{\beta} \\ \hat{\gamma} \end{bmatrix}$$

so that

$$\begin{aligned} \begin{bmatrix} \widehat{\beta} \\ \widehat{\gamma} \end{bmatrix} &= \begin{bmatrix} (1 - \zeta)S_{DD} & S_{DX} \\ (1 + \theta)S_{XD} & S_{XX} \end{bmatrix}^{-1} \begin{bmatrix} S_{YD} \\ S_{YX} \end{bmatrix} \\ &= \begin{bmatrix} \beta \\ \gamma \end{bmatrix} + \begin{bmatrix} (1 - \zeta)S_{DD} & S_{DX} \\ (1 + \theta)S_{XD} & S_{XX} \end{bmatrix}^{-1} \begin{bmatrix} S_{D\varepsilon} - \beta(S_{DU} - \zeta S_{DD}) \\ S_{X\varepsilon} - \beta(S_{XU} + \theta S_{XD}) \end{bmatrix}, \end{aligned}$$

that is,

$$\begin{bmatrix} \widehat{\beta} - \beta \\ \widehat{\gamma} - \gamma \end{bmatrix} = \begin{bmatrix} (1 - \zeta)S_{DD} & S_{DX} \\ (1 + \theta)S_{XD} & S_{XX} \end{bmatrix}^{-1} \begin{bmatrix} S_{D\varepsilon} - \beta(S_{DU} - \zeta S_{DD}) \\ S_{X\varepsilon} - \beta(S_{XU} + \theta S_{XD}) \end{bmatrix}. \quad (19)$$

Hence, taking the probability limits and applying Slutsky's lemma, we have

$$\begin{aligned} \text{plim} \begin{bmatrix} \widehat{\beta} - \beta \\ \widehat{\gamma} - \gamma \end{bmatrix} &= \text{plim} \begin{bmatrix} (1 - \zeta)S_{DD} & S_{DX} \\ (1 + \theta)S_{XD} & S_{XX} \end{bmatrix}^{-1} \text{plim} \begin{bmatrix} S_{D\varepsilon} - \beta(S_{DU} - \zeta S_{DD}) \\ S_{X\varepsilon} - \beta(S_{XU} + \theta S_{XD}) \end{bmatrix} \\ &= \Omega^{-1}\Gamma \end{aligned}$$

where, by the weak law of large numbers, we have

$$\Omega = \text{plim} \begin{bmatrix} (1 - \zeta)S_{DD} & S_{DX} \\ (1 + \theta)S_{XD} & S_{XX} \end{bmatrix} = \begin{bmatrix} (1 - \zeta)\text{Var}(D_i) & \text{Cov}(D_i, X_i) \\ (1 + \theta)\text{Cov}(X_i, D_i) & \text{Var}(X_i) \end{bmatrix} \quad (20)$$

and

$$\Gamma = \text{plim} \begin{bmatrix} S_{D\varepsilon} - \beta(S_{DU} - \zeta S_{DD}) \\ S_{X\varepsilon} - \beta(S_{XU} + \theta S_{XD}) \end{bmatrix} = \begin{bmatrix} \text{Cov}(D_i, \varepsilon_i) + \beta(\text{Cov}(D_i, U_i) - \zeta \text{Var}(D_i)) \\ \text{Cov}(X_i, \varepsilon_i) - \beta(\text{Cov}(X_i, U_i) + \theta \text{Cov}(X_i, D_i)) \end{bmatrix}.$$

But, by the L.I.E., $\mathbb{E}[\varepsilon_i | X_i, D_i] = \mathbb{E}[\mathbb{E}[\varepsilon_i | X_i, D_i^*] | X_i, D_i] = 0$, so that $\text{Cov}(D_i, \varepsilon_i) = \text{Cov}(X_i, \varepsilon_i) = 0$. Lemma 2 then implies that $\Gamma = 0$, and hence $\text{plim} \begin{bmatrix} \widehat{\beta} - \beta \\ \widehat{\gamma} - \gamma \end{bmatrix} = 0$.

This is equivalent to $\text{plim} \widehat{\beta} = \beta$ and $\text{plim} \widehat{\gamma} = \gamma$.

(ii) Asymptotic Normality. Suppose $\mathbb{E}[\varepsilon_i | X_i, D_i^*] = \sigma^2$. Then, by the L.I.E.,

$$\mathbb{E}[\varepsilon_i^2 | X_i, D_i] = \mathbb{E}[\mathbb{E}[\varepsilon_i^2 | X_i, D_i^*] | X_i, D_i] = \sigma^2.$$

Now multiplying Equation (22) above by \sqrt{n} , we have

$$\sqrt{n} \begin{bmatrix} \widehat{\beta} - \beta \\ \widehat{\gamma} - \gamma \end{bmatrix} = \begin{bmatrix} (1 - \zeta)S_{DD} & S_{DX} \\ (1 + \theta)S_{XD} & S_{XX} \end{bmatrix}^{-1} \sqrt{n} \begin{bmatrix} S_{D\varepsilon} - \beta(S_{DU} - \zeta S_{DD}) \\ S_{X\varepsilon} - \beta(S_{XU} + \theta S_{XD}) \end{bmatrix}, \quad (21)$$

which is asymptotically equivalent to

$$\sqrt{n} \begin{bmatrix} \widehat{\beta} - \beta \\ \widehat{\gamma} - \gamma \end{bmatrix} \stackrel{a}{=} \Omega^{-1} \sqrt{n} \begin{bmatrix} S_{D\varepsilon} - \beta(S_{DU} - \zeta S_{DD}) \\ S_{X\varepsilon} - \beta(S_{XU} + \theta S_{XD}) \end{bmatrix} = \Omega^{-1} \sqrt{n} \Gamma_n. \quad (22)$$

Denote:

$$Z = [D - \bar{D} \quad X - \bar{X}], \mathbf{Z} = \begin{bmatrix} D - \bar{D} & 0 \\ 0 & X - \bar{X} \end{bmatrix}, \mathbf{\Psi} = \begin{bmatrix} (U - \bar{U}) - \zeta(D - \bar{D}) \\ (U - \bar{U}) + \theta(D - \bar{D}) \end{bmatrix},$$

where Z , \mathbf{Z} , and $\mathbf{\Psi}$ are $n \times (1+k)$, $2n \times (1+k)$ and $2n \times 1$ matrices, respectively. Then

$$\sqrt{n} \Gamma_n = \sqrt{n} \begin{bmatrix} S_{D\varepsilon} - \beta(S_{DU} - \zeta S_{DD}) \\ S_{X\varepsilon} - \beta(S_{XU} + \theta S_{XD}) \end{bmatrix} = \sqrt{n} \left[\frac{Z'\varepsilon}{n} - \beta \frac{\mathbf{Z}'\mathbf{\Psi}}{n} \right]$$

By the central limit theorem, and given that the vector ε is asymptotically uncorrelated with the components of $\mathbf{Z}'\mathbf{\Psi}$, the asymptotic variance of Γ_n is then given by

$$\Xi = \sigma^2 \text{plim} \frac{Z'Z}{n} + \beta^2 \text{plim} \frac{\mathbf{Z}'\mathbf{\Sigma}_{\mathbf{\Psi}}\mathbf{Z}}{n}$$

with

$$\mathbf{\Sigma}_{\mathbf{\Psi}} = \begin{bmatrix} \text{Var}(U_i) - \zeta^2 \text{Var}(D_i) & \text{Var}(U_i) - \zeta^2 \text{Var}(D_i) \\ \text{Var}(U_i) - \zeta^2 \text{Var}(D_i) & \text{Var}(U_i) + (2\theta\zeta + \theta^2) \text{Var}(D_i) \end{bmatrix} \otimes \mathbf{I}_n = \begin{bmatrix} \kappa \mathbf{I}_n & \kappa \mathbf{I}_n \\ \kappa \mathbf{I}_n & \tilde{\kappa} \mathbf{I}_n \end{bmatrix}$$

where

$$\kappa = \text{Var}(U_i) - \zeta^2 \text{Var}(D_i), \quad \text{and} \quad \tilde{\kappa} = \kappa + (\zeta + \theta)^2 \text{Var}(D_i)$$

Noticing that $\text{Var}(U_i) = (\zeta - \theta(1 - \zeta)) \text{Var}(D_i)$, we then have:

$$\Xi = \begin{bmatrix} (\sigma^2 + \beta^2 \kappa) \text{Var}(D_i) & (\sigma^2 + \beta^2 \kappa) \text{Cov}(D_i, X_i) \\ (\sigma^2 + \beta^2 \kappa) \text{Cov}(X_i, D_i) & (\sigma^2 + \beta^2 \tilde{\kappa}) \text{Var}(X_i) \end{bmatrix}. \quad (23)$$

where:

$$\kappa = (1 - \zeta)(\zeta - \theta) \text{Var}(D_i), \quad \text{and} \quad \tilde{\kappa} = [(1 - \zeta)(\zeta - \theta) + (\zeta + \theta)^2] \text{Var}(D_i) \quad (24)$$

We therefore conclude that $\sqrt{n} \begin{bmatrix} \widehat{\beta} - \beta \\ \widehat{\gamma} - \gamma \end{bmatrix} \xrightarrow{d} N(\mathbf{0}, \mathbf{\Sigma})$, with

$$\mathbf{\Sigma} = \Omega^{-1} \Xi \Omega^{-1} \quad (25)$$

where the matrices Ω , and Ξ are given by equations (20), (23) and (24). All the components of the matrix $\mathbf{\Sigma}$ can be easily estimated from sample information and knowledge of the misclassification probabilities. Specifically, these quantities are

estimated using their sample counterparts and by plugging in the BALS estimates $\hat{\beta}$ and $\hat{\gamma}$, as well as the residual variance $\hat{\sigma}^2$ defined by

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n (Y_i - \hat{Y}_i)^2, \quad \text{where} \quad \hat{Y}_i = \hat{c} + \hat{\beta} \frac{D_i - \alpha_0}{1 - \alpha_0 - \alpha_1} + X_i' \hat{\gamma}. \quad (26)$$

Now, noticing that $\sqrt{n}(\hat{\beta} - \beta) = e'_{(1)} \sqrt{n} \begin{bmatrix} \hat{\beta} - \beta \\ \hat{\gamma} - \gamma \end{bmatrix}$ and $\sqrt{n}(\hat{\gamma} - \gamma) = [0 \ I_k] \sqrt{n} \begin{bmatrix} \hat{\beta} - \beta \\ \hat{\gamma} - \gamma \end{bmatrix}$,

where $e_{(1)} = [1 \ 0 \dots 0]'$ is a $(k+1) \times 1$ vector and I_k is the identity matrix of size k , we can finally derive the asymptotic variances of $\hat{\beta}$ and $\hat{\gamma}$ from Σ by defining

$$\sigma_{\beta}^2 = e'_{(1)} \Sigma e_{(1)}, \quad \text{and} \quad \Sigma_{\gamma} = [0 \ I_k] \Sigma \begin{bmatrix} 0 \\ I_k \end{bmatrix} \quad (27)$$

Then, by Slutsky's lemma, we have:

$$\sqrt{n}(\hat{\beta} - \beta) \xrightarrow{d} N(0, \sigma_{\beta}^2), \quad \text{and} \quad \sqrt{n}(\hat{\gamma} - \gamma) \xrightarrow{d} N(0, \Sigma_{\gamma})$$

□

7.6 Proof of Corollary 2

Proof. To prove the consistency of \hat{c} , we note that $\bar{Y} = c + \beta \bar{D}^* + \bar{X}' \gamma + \bar{\varepsilon}$. Then

$$\begin{aligned} \hat{c} - c &= \beta \bar{D}^* - \hat{\beta} \frac{\bar{D} - \alpha_0}{1 - \alpha_0 - \alpha_1} + \bar{X}'(\gamma - \hat{\gamma}) + \bar{\varepsilon} \\ &= \beta \left(\bar{D}^* - \frac{\bar{D} - \alpha_0}{1 - \alpha_0 - \alpha_1} \right) + (\beta - \hat{\beta}) \frac{\bar{D} - \alpha_0}{1 - \alpha_0 - \alpha_1} + \bar{X}'(\gamma - \hat{\gamma}) + \bar{\varepsilon} \end{aligned}$$

Then taking the probability limits and using the the law of large numbers, we have

$$\begin{aligned} \text{plim}(\hat{c} - c) &= \beta \left(P^* - \frac{P - \alpha_0}{1 - \alpha_0 - \alpha_1} \right) + \text{plim}(\hat{\beta} - \beta) \frac{P - \alpha_0}{1 - \alpha_0 - \alpha_1} \\ &\quad + \mathbb{E}[X_i]' \text{plim}(\gamma - \hat{\gamma}) + \mathbb{E}[\varepsilon_i] \\ &= 0, \end{aligned}$$

by the consistency of $\hat{\beta}$ and $\hat{\gamma}$ and the relationship between the true and the reported mean responses.

For the asymptotic normality, we write

$$\sqrt{n}(\hat{c}-c) = \begin{bmatrix} \frac{\bar{D} - \alpha_0}{1 - \alpha_0 - \alpha_1} & \bar{X}' \end{bmatrix} \sqrt{n} \begin{bmatrix} \hat{\beta} - \beta \\ \hat{\gamma} - \gamma \end{bmatrix} + [\beta \quad 1] \sqrt{n} \begin{bmatrix} \bar{D}^* - \frac{\bar{D} - \alpha_0}{1 - \alpha_0 - \alpha_1} \\ \bar{\varepsilon} \end{bmatrix}.$$

Then, using the asymptotic variance of $[\hat{\beta}, \hat{\gamma}]'$ derived above, Σ , and the independence of ε_i with respect to D_i^* , D_i and X_i , the asymptotic variance of \hat{c} is obtained as

$$\sigma_c^2 = H' \Sigma H + \frac{\beta^2(\theta + \zeta)}{1 - \alpha_0 - \alpha_1} \text{Var}(D_i) + \sigma^2 \quad (28)$$

where $H = \begin{bmatrix} \frac{P - \alpha_0}{1 - \alpha_0 - \alpha_1} & \mathbb{E}[X_i]' \end{bmatrix}'$ is the vector of mean regressors from the true model. As above, this variance can be estimated using sample information and knowledge of the misclassification probabilities. \square

7.7 Proof of Theorem 3

Proof. Recall that the BALS Estimator is given by

$$\begin{bmatrix} \hat{\beta} \\ \hat{\gamma} \end{bmatrix} = \begin{bmatrix} (1 - \zeta)S_{DD} & S_{DX} \\ (1 + \theta)S_{XD} & S_{XX} \end{bmatrix}^{-1} \begin{bmatrix} S_{YD} \\ S_{YX} \end{bmatrix} = \Omega_n(\boldsymbol{\alpha})^{-1} \begin{bmatrix} S_{YD} \\ S_{YX} \end{bmatrix}$$

where $\boldsymbol{\alpha} = (\alpha_0, \alpha_1)$ and $\Omega_n(\boldsymbol{\alpha}) = \begin{bmatrix} (1 - \zeta)S_{DD} & S_{DX} \\ (1 + \theta)S_{XD} & S_{XX} \end{bmatrix}$.

In the FBALS estimation, we replace the unknown matrix $\Omega_n(\boldsymbol{\alpha})$ with a consistent estimator $\Omega_n(\hat{\boldsymbol{\alpha}})$, that is:

$$\begin{bmatrix} \hat{\hat{\beta}} \\ \hat{\hat{\gamma}} \end{bmatrix} = \begin{bmatrix} (1 - \hat{\zeta})S_{DD} & S_{DX} \\ (1 + \hat{\theta})S_{XD} & S_{XX} \end{bmatrix}^{-1} \begin{bmatrix} S_{YD} \\ S_{YX} \end{bmatrix} = \Omega_n(\hat{\boldsymbol{\alpha}})^{-1} \begin{bmatrix} S_{YD} \\ S_{YX} \end{bmatrix}$$

where $\hat{\boldsymbol{\alpha}} = (\hat{\alpha}_0, \hat{\alpha}_1)$ is the estimator of $\boldsymbol{\alpha} = (\alpha_0, \alpha_1)$ obtained in the first-step.

The asymptotic properties of the FBALS given by part (i) and part(ii) of the theorem, it is sufficient to show that

$$\text{plim} \begin{bmatrix} \hat{\hat{\beta}} - \hat{\beta} \\ \hat{\hat{\gamma}} - \hat{\gamma} \end{bmatrix} = 0 \quad (29)$$

Then by Slutsky's Theorem, it will readily follow, given Theorem 2, that $\text{plim} \begin{bmatrix} \hat{\hat{\beta}} \\ \hat{\hat{\gamma}} \end{bmatrix} = \begin{bmatrix} \beta \\ \gamma \end{bmatrix}$ and that $\text{plim} \begin{bmatrix} \hat{\hat{\beta}} \\ \hat{\hat{\gamma}} \end{bmatrix}$ has the same asymptotic distribution

as $\begin{bmatrix} \hat{\beta} \\ \hat{\gamma} \end{bmatrix}$. Given the continuity of the inverse function and Assumptions 1-2, it is sufficient to show that $\text{plim} [\Omega_n(\hat{\alpha}) - \Omega_n(\alpha)] = \mathbf{0}$ for Equation (29) to hold.

$$\begin{aligned} \text{plim} [\Omega_n(\hat{\alpha}) - \Omega_n(\alpha)] &= \text{plim} \begin{bmatrix} (\zeta - \hat{\zeta})S_{DD} & S_{DX} \\ (\hat{\theta} - \theta)S_{XD} & S_{XX} \end{bmatrix} \\ &= \begin{bmatrix} \text{plim}(\zeta - \hat{\zeta})\text{Var}(D_i) & \text{Cov}(D_i, X_i) \\ \text{plim}(\hat{\theta} - \theta)\text{Cov}(X_i, D_i) & \text{Var}(X_i) \end{bmatrix} = \mathbf{0} \end{aligned}$$

where the last two equalities follow from the law of large numbers, the continuous mapping theorem, and the consistency $(\hat{\alpha}_0, \hat{\alpha}_1)$. \square

7.8 Proof of Theorem 4

We can write $\tilde{D}_i = (1 - \alpha_0 - \alpha_1)\tilde{D}_i^* + \nu_i$, where $\mathbb{E}[\nu_i|X_i, D_i^*] = 0$. The residual model given in (17) can also be rewritten as

$$\tilde{Y}_i = \delta\tilde{D}_i^{**} + \varepsilon_i, \quad \mathbb{E}[\varepsilon_i|\tilde{D}_i^{**}] = 0 \quad (30)$$

where $\delta = \frac{\beta}{1 - \alpha_0 - \alpha_1}$ and $\tilde{D}_i^{**} = (1 - \alpha_0 - \alpha_1)\tilde{D}_i^*$.

Our bounding approach proceeds as follows. We start by bounding the regression slope δ from the residual model (30). The quantity δ is an index of the amount of measurement error in the system, since for a given β , a larger δ implies larger values of $\alpha_0 + \alpha_1$. For a given amount of measurement error δ , the possible values of α_0 and α_1 are determined. Since $\beta = (1 - \alpha_0 - \alpha_1)\delta$, the set of all possible values of β given all feasible values of δ , α_0 and α_1 can then be derived as the identifying set.

Lemma 3. *Under Assumptions 1-2, we have*

$$b \leq \delta \leq \min \left\{ \frac{\sigma_0^2(1-P)^2}{\sigma_{\tilde{D}\tilde{Y}}(1-\alpha_0)\alpha_1} \left[1 - \frac{P(1-P)R_{DX}^2}{(P-\alpha_0)(1-\alpha_1-P)} \right], \frac{\sigma_1^2 P^2}{\sigma_{\tilde{D}\tilde{Y}}(1-\alpha_1)\alpha_0} \left[1 - \frac{P(1-P)R_{DX}^2}{(P-\alpha_0)(1-\alpha_1-P)} \right] \right\}$$

Proof. The conditional variance of Y_i given X_i and $D_i = 0$ can be obtained as

$$\begin{aligned}
\sigma_0^2 &= \text{Var}[\tilde{Y}_i | X_i, D_i = 0] = \delta^2 \text{Var}[\tilde{D}_i^{**} | X_i, D_i = 0] + \text{Var}[\varepsilon_i | X_i, D_i = 0] \\
&= \delta^2 (1 - \alpha_0 - \alpha_1)^2 \text{Var}[\tilde{D}_i^* | X_i, D_i = 0] + \text{Var}[\varepsilon_i | X_i, D_i = 0] \\
&= \delta^2 (1 - \alpha_0 - \alpha_1)^2 \frac{\text{Var}[D_i^*] \alpha_1 (1 - \alpha_0)}{(1 - P)^2} + \text{Var}[\varepsilon_i | X_i, D_i = 0] \\
&= \delta^2 (1 - \alpha_0 - \alpha_1)^2 \frac{\text{Var}[\tilde{D}_i^*] \alpha_1 (1 - \alpha_0)}{(1 - R_{D^*X}^2)(1 - P)^2} + \text{Var}[\varepsilon_i | X_i, D_i = 0] \\
&= \delta^2 \frac{\text{Var}[\tilde{D}_i^{**}] \alpha_1 (1 - \alpha_0)}{(1 - R_{D^*X}^2)(1 - P)^2} + \text{Var}[\varepsilon_i | X_i, D_i = 0]
\end{aligned}$$

Now, notice that $\text{Cov}(\tilde{D}_i, \tilde{Y}_i) = \delta \text{Var}[\tilde{D}_i^{**}]$. Hence

$$\sigma_0^2 = \delta \frac{\sigma_{\tilde{D}\tilde{Y}} \alpha_1 (1 - \alpha_0)}{(1 - R_{D^*X}^2)(1 - P)^2} + \text{Var}[\varepsilon_i | X_i, D_i = 0]. \text{ Since } \text{Var}[\varepsilon_i | X_i, D_i = 0] \geq 0,$$

we then have $\sigma_0^2 \geq \delta \frac{\sigma_{\tilde{D}\tilde{Y}} \alpha_1 (1 - \alpha_0)}{(1 - R_{D^*X}^2)(1 - P)^2}$, so that $\delta \leq \frac{\sigma_0^2 (1 - P)^2}{\sigma_{\tilde{D}\tilde{Y}} (1 - \alpha_0) \alpha_1} (1 - R_{D^*X}^2)$.

Finally, given the results in Lemma 1, we show that

$$R_{D^*X}^2 = \frac{\text{Cov}(D^*, X) \text{Var}(X)^{-1} \text{Cov}(X, D^*)}{\text{Var}(D^*)} = \frac{P(1 - P)R_{DX}^2}{(P - \alpha_0)(1 - \alpha_1 - P)}.$$

It follows that

$$\delta \leq \frac{\sigma_0^2 (1 - P)^2}{\sigma_{\tilde{D}\tilde{Y}} (1 - \alpha_0) \alpha_1} \left[1 - \frac{P(1 - P)R_{DX}^2}{(P - \alpha_0)(1 - \alpha_1 - P)} \right] \quad (31)$$

Likewise, the conditional variance of Y_i given X_i and $D_i = 1$ can be obtained as

$$\begin{aligned}
\sigma_1^2 &= \text{Var}[\tilde{Y}_i | X_i, D_i = 1] = \delta^2 \text{Var}[\tilde{D}_i^{**} | X_i, D_i = 1] + \text{Var}[\varepsilon_i | X_i, D_i = 1] \\
&= \delta^2 (1 - \alpha_0 - \alpha_1)^2 \text{Var}[\tilde{D}_i^* | X_i, D_i = 0] + \text{Var}[\varepsilon_i | X_i, D_i = 1] \\
&= \delta^2 (1 - \alpha_0 - \alpha_1)^2 \frac{\text{Var}[D_i^*] \alpha_0 (1 - \alpha_1)}{P^2} + \text{Var}[\varepsilon_i | X_i, D_i = 1] \\
&= \delta^2 (1 - \alpha_0 - \alpha_1)^2 \frac{\text{Var}[\tilde{D}_i^*] \alpha_1 (1 - \alpha_0)}{(1 - R_{D^*X}^2)P^2} + \text{Var}[\varepsilon_i | X_i, D_i = 1] \\
&= \delta^2 \frac{\text{Var}[\tilde{D}_i^{**}] \alpha_0 (1 - \alpha_1)}{(1 - R_{D^*X}^2)P^2} + \text{Var}[\varepsilon_i | X_i, D_i = 1] \\
&= \delta \frac{\sigma_{\tilde{D}\tilde{Y}} \alpha_0 (1 - \alpha_1)}{(1 - R_{D^*X}^2)P^2} + \text{Var}[\varepsilon_i | X_i, D_i = 1]
\end{aligned}$$

Since $\text{Var}[\varepsilon_i|X_i, D_i = 1] \geq 0$, we then have $\sigma_1^2 \geq \delta \frac{\sigma_{\tilde{D}\tilde{Y}}\alpha_0(1-\alpha_1)}{(1-R_{D^*X}^2)P^2}$, so that $\delta \leq \frac{\sigma_1^2 P^2}{\sigma_{\tilde{D}\tilde{Y}}(1-\alpha_1)\alpha_0}(1-R_{D^*X}^2)$. Plugging in the expression of $R_{D^*X}^2$ obtained above, we also have

$$\delta \leq \frac{\sigma_1^2 P^2}{\sigma_{\tilde{D}\tilde{Y}}(1-\alpha_1)\alpha_0} \left[1 - \frac{P(1-P)R_{DX}^2}{(P-\alpha_0)(1-\alpha_1-P)} \right] \quad (32)$$

From inequalities (32) and (32), we must then have

$$\delta \leq \min \left\{ \frac{\sigma_0^2(1-P)^2}{\sigma_{\tilde{D}\tilde{Y}}(1-\alpha_0)\alpha_1} \left[1 - \frac{P(1-P)R_{DX}^2}{(P-\alpha_0)(1-\alpha_1-P)} \right], \frac{\sigma_1^2 P^2}{\sigma_{\tilde{D}\tilde{Y}}(1-\alpha_1)\alpha_0} \left[1 - \frac{P(1-P)R_{DX}^2}{(P-\alpha_0)(1-\alpha_1-P)} \right] \right\} \quad (33)$$

This gives us an upper bound for δ . The lower bound is obtained by noticing that $\text{Var}(\tilde{D}_i) = \text{Var}(\tilde{D}_i^{**}) + \text{Var}(\nu_i) = \frac{\text{Cov}(\tilde{D}, \tilde{Y})}{\delta} + \text{Var}(\nu_i)$. That is, $\text{Var}(\nu_i) = \sigma_{\tilde{D}}^2 - \frac{\sigma_{\tilde{D}\tilde{Y}}}{\delta}$. Since $\text{Var}(\nu_i) \geq 0$, we then have $\sigma_{\tilde{D}}^2 \geq \frac{\sigma_{\tilde{D}\tilde{Y}}}{\delta}$, so that

$$\delta \geq \frac{\sigma_{\tilde{D}\tilde{Y}}}{\sigma_{\tilde{D}}^2} = b \quad (34)$$

□

Lemma 4. *Given Model (1) and any feasible value of δ from Lemma 3, we have*

$$\alpha_1 = (1-P) \left\{ 1 - \left(\frac{P}{P-\alpha_0} \right) \left[(1-R_{DX}^2) \frac{b}{\delta} + R_{DX}^2 \right] \right\} \quad (35)$$

Proof. Given that $\tilde{D}_i = (1-\alpha_0-\alpha_1)\tilde{D}_i^* + \nu_i = \tilde{D}_i^{**} + \nu_i$, we have, on the one hand,

$$\begin{aligned} \text{Var}(\tilde{D}_i) &= (1-\alpha_0-\alpha_1)^2 \text{Var}(\tilde{D}_i^*) + \text{Var}(\nu_i) \\ &= (1-\alpha_0-\alpha_1)^2 \text{Var}(D_i^*)(1-R_{D^*X}^2) + \text{Var}(\nu_i) \\ &= (P-\alpha_0)(1-\alpha_1-P)(1-R_{D^*X}^2) + \text{Var}(\nu_i) \end{aligned}$$

so that

$$\begin{aligned} \text{Var}(\nu_i) &= \text{Var}(\tilde{D}_i) - (P-\alpha_0)(1-\alpha_1-P)(1-R_{D^*X}^2) \\ &= \text{Var}(\tilde{D}_i) - (P-\alpha_0)(1-\alpha_1-P) \left[1 - \frac{P(1-P)R_{DX}^2}{(P-\alpha_0)(1-\alpha_1-P)} \right] \\ &= \text{Var}(\tilde{D}_i) - (P-\alpha_0)(1-\alpha_1-P) + P(1-P)R_{DX}^2 \end{aligned} \quad (36)$$

On the other hand, $\text{Var}(\tilde{D}_i) = \text{Var}(\tilde{D}_i^{**}) + \text{Var}(\nu_i) = \frac{\sigma_{\tilde{D}\tilde{Y}}}{\delta} + \text{Var}(\nu_i)$ so that

$$\text{Var}(\nu_i) = \text{Var}(\tilde{D}_i) - \frac{\sigma_{\tilde{D}\tilde{Y}}}{\delta} = \text{Var}(\tilde{D}_i) \left[1 - \frac{b}{\delta} \right] = \text{Var}(\tilde{D}_i) - \frac{b}{\delta} P(1-P)(1-R_{DX}^2) \quad (37)$$

Equalizing (36) and (37) yields

$$(P - \alpha_0)(1 - \alpha_1 - P) - P(1-P)R_{DX}^2 = \frac{b}{\delta} P(1-P)(1 - R_{DX}^2)$$

Solving for α_1 then gives the desired result. \square

Proof of Part (i) of Theorem 4

Denote $\tau = 1 - \alpha_0 - \alpha_1$ so that we can write $\beta = \tau\delta$. The maximum value of τ occurs when $\alpha_0 = 0$ or $\alpha_1 = 0$.

If $P > 1/2$, then the upper bound is associated with the case where $\alpha_0 = 0$. In this case, the value of α_1 is given by

$$\alpha_1^m = (1-P)(1-R_{DX}^2) \left(1 - \frac{b}{\delta^{max,0}} \right)$$

where $\delta^{max,0}$ denotes the corresponding value of δ . The inequality in Lemma 3, then reduces to

$$b \leq \delta \leq \frac{\sigma_0^2(1-P)^2}{\sigma_{\tilde{D}\tilde{Y}}\alpha_1^m} \left(1 - \frac{(1-P)R_{DX}^2}{(1-\alpha_1^m - P)} \right) \leq \frac{\sigma_0^2(1-P)^2}{\sigma_{\tilde{D}\tilde{Y}}\alpha_1^m} (1 - R_{DX}^2)$$

This implies that $\delta^{max,0}$ and α_1^m are jointly determined by the system:

$$\delta^{max,0} = \frac{\sigma_0^2(1-P)^2}{\sigma_{\tilde{D}\tilde{Y}}\alpha_1^m} (1 - R_{DX}^2), \quad \alpha_1^m = (1-P)(1 - R_{DX}^2) \left(1 - \frac{b}{\delta^{max,0}} \right)$$

which yields

$$\delta^{max,0} = b + \frac{\sigma_0^2(1-P)}{\sigma_{\tilde{D}\tilde{Y}}} = b + \kappa_0 \quad \text{and} \quad \alpha_1^m = \frac{(1-P)^2\sigma_0^2(1 - R_{DX}^2)}{b\sigma_{\tilde{D}\tilde{Y}} + \sigma_0^2(1-P)} \quad (38)$$

Then, the maximum value of β when $P > 1/2$ is given by

$$\begin{aligned} \beta^{max,0} &= \tau^{max,0} \delta^{max,0} = (1 - \alpha_1^m) \delta^{max,0} = \delta^{max,0} - \alpha_1^m \delta^{max,0} \\ &= \delta^{max,0} - (1-P)(1 - R_{DX}^2) (\delta^{max,0} - b) \\ &= \delta^{max,0} [1 - (1-P)(1 - R_{DX}^2)] + b(1-P)(1 - R_{DX}^2) \\ &= (b + \kappa_0) [1 - (1-P)(1 - R_{DX}^2)] + b(1-P)(1 - R_{DX}^2) \end{aligned} \quad (39)$$

If $P < 1/2$, then the upper bound is associated with the case where $\alpha_1 = 0$. Using Lemma 4 the value of α_0 is given by

$$\alpha_0^m = P(1 - R_{DX}^2) \left(1 - \frac{b}{\delta^{max,1}} \right)$$

where $\delta^{max,1}$ denotes the corresponding value of δ . The inequality in Lemma 3, then reduces to

$$b \leq \delta \leq \frac{\sigma_1^2 P^2}{\sigma_{\tilde{D}\tilde{Y}} \alpha_0} \left(1 - \frac{PR_{DX}^2}{(P - \alpha_0)} \right) \leq \frac{\sigma_1^2 P^2}{\sigma_{\tilde{D}\tilde{Y}} \alpha_0} (1 - R_{DX}^2)$$

This implies that $\delta^{max,1}$ and α_0^m are jointly determined by the system:

$$\delta^{max,1} = \frac{\sigma_1^2 P^2}{\sigma_{\tilde{D}\tilde{Y}} \alpha_0^m} (1 - R_{DX}^2), \quad \alpha_0^m = P(1 - R_{DX}^2) \left(1 - \frac{b}{\delta^{max,1}} \right)$$

which yields

$$\delta^{max,1} = b + \frac{\sigma_1^2 P}{\sigma_{\tilde{D}\tilde{Y}}} = b + \kappa_1 \quad \text{and} \quad \alpha_0^m = \frac{P^2 \sigma_1^2 (1 - R_{DX}^2)}{b \sigma_{\tilde{D}\tilde{Y}} + \sigma_0^2 P} \quad (40)$$

Hence, the maximum value of β when $P < 1/2$ is given by

$$\begin{aligned} \beta^{max,1} &= \tau^{max,1} \delta^{max,1} = (1 - \alpha_0^m) \delta^{max,1} = \delta^{max,1} - \alpha_0^m \delta^{max,1} \\ &= \delta^{max,1} - P(1 - R_{DX}^2) (\delta^{max,1} - b) \\ &= \delta^{max,1} [1 - P(1 - R_{DX}^2)] + bP(1 - R_{DX}^2) \\ &= (b + \kappa_1) [1 - P(1 - R_{DX}^2)] + bP(1 - R_{DX}^2) \end{aligned} \quad (41)$$

It follows that δ is bounded by

$$b \leq \delta \leq \delta^{max,0} \mathbf{1}[P > 1/2] + \delta^{max,1} \mathbf{1}[P < 1/2] = b + \kappa, \quad (42)$$

where $\kappa_0 = \frac{\sigma_0^2(1 - P)}{\sigma_{\tilde{D}\tilde{Y}}}$, $\kappa_1 = \frac{\sigma_1^2 P}{\sigma_{\tilde{D}\tilde{Y}}}$, and $\kappa = \kappa_0 \mathbf{1}[P > 1/2] + \kappa_1 \mathbf{1}[P \leq 1/2]$. Likewise, β is bounded by $b \leq \beta \leq \beta^{max} = \max\{\beta^{max,0}, \beta^{max,1}\}$, where $\beta^{max,0}$ and $\beta^{max,1}$ are given by Equations (38) and (40), respectively.

Proof of Part (ii) of Theorem 4

Notice that $\psi = \text{Var}(X_i)^{-1} \text{Cov}(X_i, Y_i) = \text{Var}(X_i)^{-1} \text{Cov}(X_i, c + \beta D_i^* + X_i' \gamma + \varepsilon_i) = \beta \text{Var}(X_i)^{-1} \text{Cov}(X_i, D_i^*) + \gamma = \delta \text{Var}(X_i)^{-1} \text{Cov}(X_i, D_i) + \gamma = \delta \lambda + \gamma$.

That is,

$$\gamma = \psi - \delta \lambda. \quad (43)$$

Hence, $\gamma_j = \psi_j - \delta\lambda_j$, $j = 1, \dots, k$. If $\lambda_j \geq 0$, then by the inequality $b \leq \delta \leq b + \kappa$ obtained from Expression (42) above, we must have $\psi_j - b\lambda_j - \kappa\lambda_j \leq \gamma_j \leq \psi_j - b\lambda_j$. Otherwise, if $\lambda_j \leq 0$, then we must instead have $\psi_j - b\lambda_j \leq \gamma_j \leq \psi_j - b\lambda_j - \kappa\lambda_j$.

Proof of Part (iii) of Theorem 4

We can write:

$$\begin{aligned}
\psi_0 - \delta\lambda_0 &= E[Y] - E[X]'\psi - \delta E[D] + \delta E[X]'\lambda \\
&= E[Y] - E[X]'\text{Var}(X)^{-1}\text{Cov}(X, Y) - \delta E[D] + \delta E[X]'\text{Var}(X)^{-1}\text{Cov}(X, D) \\
&= E[Y] - \beta E[D^*] + \delta\alpha_0 - E[X]'\text{Var}(X)^{-1}\text{Cov}(X, c + \beta D^* + X'\gamma + \varepsilon) \\
&\quad + \beta E[X]'\text{Var}(X)^{-1}\text{Cov}(X, D^*) \\
&= c + E[X]'\gamma - \delta\alpha_0 - \beta E[X]'\text{Var}(X)^{-1}\text{Cov}(X, D^*) - E[X]'\gamma \\
&\quad + \beta E[X]'\text{Var}(X)^{-1}\text{Cov}(X, D^*) \\
&= c - \delta\alpha_0
\end{aligned}$$

That is

$$c = \psi_0 - \delta\lambda_0 + \delta\alpha_0 \geq \psi_0 - \delta\lambda_0$$

The lower bound of c is then $c_{min} = \psi_0 - \delta\lambda_0$. Since $b \leq \delta \leq b + \kappa$, this means $c_{min} = \min\{\psi_0 - b\lambda_0, \psi_0 - (b + \kappa)\lambda_0\}$.

We obtain the upper bound of c as follows. We know from the preceding discussions that the value of α_0 associated with any upper bound (i.e. whether $P > 1/2$ or $P \leq 1/2$) is such that $0 \leq \alpha_0 \leq P(1 - R_{DX}^2) \left(1 - \frac{b}{\delta}\right)$, where $b \leq \delta \leq \delta^{max,1} = b + \kappa_1$. We then have

$$\begin{aligned}
c &= \psi_0 - \delta\lambda_0 + \delta\alpha_0 \leq \psi_0 - \delta\lambda_0 + P(1 - R_{DX}^2)(\delta - b) \\
&= \psi_0 - \delta [\lambda_0 - P(1 - R_{DX}^2)] - bP(1 - R_{DX}^2)
\end{aligned}$$

If $\lambda_0 - P(1 - R_{DX}^2) \leq 0$, then $c_{max} = \psi_0 - b[\lambda_0 - P(1 - R_{DX}^2)] - bP(1 - R_{DX}^2) = \psi_0 - b\lambda_0$

If $\lambda_0 - P(1 - R_{DX}^2) > 0$, then $c_{max} = \psi_0 - (b + \kappa_1)[\lambda_0 - P(1 - R_{DX}^2)] - bP(1 - R_{DX}^2) = \psi_0 - (b + \kappa_1)\lambda_0 + P(1 - R_{DX}^2)\kappa_1$.

Proof of Part (iv) of Theorem 4

The upper bounds of α_0 and α_1 are given by α_0^m and α_1^m in Equations (40) and (38), respectively. \square

Proof that the bounds in Theorem 4 are tighter than the bounds in Bollinger (1996, Theorem 5)

Notice that the bounds in Theorem 4 above have expressions similar to the bounds in Bollinger (1996, Theorem 5), except that in the former κ_0 or κ_1 are used in lieu of $d - b$ in the latter, where $d = \frac{\text{Var}(\tilde{Y}_i)}{\text{Cov}(\tilde{Y}_i, \tilde{D}_i)}$ is the inverse of the OLS coefficient of the regression of \tilde{Y}_i on \tilde{D}_i .

We therefore only need to show that $b + \kappa_0 < d$ and $b + \kappa_1 < d$ to conclude. We know, from Equations (38) and (40), that this is equivalent to showing that $\delta^{\max,0} < d$ and $\delta^{\max,1} < d$, where $\delta^{\max,0}$ is the value of δ associated with $(\alpha_0, \alpha_1) = (0, \alpha_1^m)$ and $\delta^{\max,1}$ is the value of δ associated with $(\alpha_0, \alpha_1) = (\alpha_0^m, 0)$.

To see why these inequalities hold, notice that from Equation 30, i.e. $\tilde{Y}_i = \delta \tilde{D}_i^{**} + \varepsilon_i$, we have

$$\text{Var}(\tilde{Y}_i) = \delta^2 \text{Var}(\tilde{D}_i^{**}) + \text{Var}(\varepsilon_i) = \delta \text{Cov}(\tilde{Y}_i, \tilde{D}_i) + \text{Var}(\varepsilon_i) \quad (44)$$

Therefore $\text{Var}(\varepsilon_i) > 0$ implies $\text{Var}(\tilde{Y}_i) - \delta \text{Cov}(\tilde{Y}_i, \tilde{D}_i) > 0$, i.e. $\delta < \frac{\text{Var}(\tilde{Y}_i)}{\text{Cov}(\tilde{Y}_i, \tilde{D}_i)} = d$, and this inequality is true for any couplet of (α_0, α_1) . Hence, it holds for the particular cases $\delta^{\max,0}$ and $\delta^{\max,1}$ of δ . \square

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